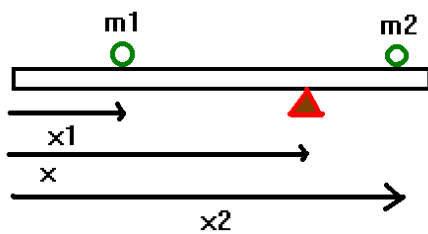


(1) Suppose a force,  $\vec{F}=F\hat{j}$  is applied to a solid wheel of mass  $M$  and radius  $R$  at the edge and at the position  $\vec{R}=R\hat{i}$  which is at right angles to the direction of the force. The magnitude of the force varies with time as  $F=ct$  [in the SI system,  $c$  has units of N/s]. Find the angular velocity and angular position at some later time  $t$  if the wheel was initially at rest and initially at  $\theta=0$ . Discuss the additional cases where  $\alpha=-h\theta$  and  $\alpha=b\omega$  where  $a$  and  $b$  are constants.

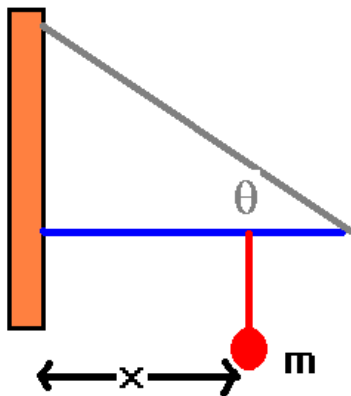
(2) A meter stick of mass  $m=0.05$  kg is placed on a pivot at  $x=0.7$  m. A mass  $m_1=0.5$  kg is placed at  $x=0.3$  m. Where would a mass  $m_2=0.8$  kg need to be placed in order to balance the system?



(3) Solve problem 2 considering the axis to be at the center of mass (Not! The pivot).

(4) Show that a meter stick which is in equilibrium about one axis is in equilibrium about all axes.

(5) A rod of mass  $M$  and length  $L$  is attached to a wall as shown. A mass  $m$  is placed on the rod at a distance  $x$  from the wall. Find the tension in the cable, and the forces on the wall.



(1) Suppose a force,  $\vec{F}=F\hat{j}$  is applied to a solid wheel of mass  $M$  and radius  $R$  at the edge and at the position  $\vec{R}=R\hat{i}$  which is at right angles to the direction of the force. The magnitude of the force varies with time as  $F=ct$  [in the SI system,  $c$  has units of N/s]. Find the angular velocity and angular position at some later time  $t$  if the wheel was initially at rest and initially at  $\theta=0$ . Discuss the additional cases where  $\alpha=-h\theta$  and  $\alpha=b\omega$  where  $a$  and  $b$  are constants.

Solution: In general, the angular acceleration is defined by:

$$\alpha \equiv \frac{d\omega}{dt} \Rightarrow d\omega = \alpha dt$$

$$\int_0^{\omega} d\omega = \int_0^t \alpha dt \Rightarrow \omega = \int_0^t \alpha dt$$

If the angular acceleration is a constant, we then see  $\omega = \alpha t$  but here, we are given that the force applied varies with time as  $ct$ . In order to find out what the angular acceleration is, we need to find the torque. Thus,

$$\vec{\Gamma} = \vec{R} \times \vec{F} = RF(\hat{i} \times \hat{j}) = RF\hat{k}$$

This torque would be considered to be in the positive direction. The magnitude of the torque is given by:

$$|\vec{\Gamma}| = RF = Rct$$

Now that we have the torque, we can find  $\alpha$ . First, we'll also need the moment of inertia which, for a solid disk rotating about its central axis is given by  $I = \frac{1}{2}mR^2$ . Thus, to find  $\alpha$  we use:

$$(\Gamma = Rct) = \left( I\alpha = \frac{1}{2}mR^2\alpha \right) \Rightarrow \alpha = \frac{2Rct}{m} R^2 = \frac{2ct}{mR}$$

We thus can find the angular velocity as:

$$\omega = \int_0^t \alpha dt = \int_0^t \frac{2ct}{mR} dt = 2 \frac{c}{mR} \int_0^t t dt = 2 \frac{c}{mR} \frac{t^2}{2} = \frac{ct^2}{mR}$$

Now, in order to find the position as a function of time, we need to once again apply the definition of angular velocity:

$$\omega = \frac{d\theta}{dt} \Rightarrow d\theta = \omega dt \Rightarrow \int_0^{\theta} d\theta = \int_0^t \omega dt \Rightarrow \theta = \int_0^t dt$$

If we had an angular velocity which varied linearly with time (i.e. a constant angular acceleration), then we would get the familiar equation of motion for  $\theta$ :  $\theta = \frac{1}{2}\omega t^2$ . Here,

however, we've seen that this is not the case, indeed,  $\omega = \frac{ct^2}{mR}$ . Thus, we find  $\theta$  as:

$$\theta = \int_0^t \omega dt = \int_0^t \frac{ct^2}{mR} dt = \frac{c}{mR} \int_0^t t^2 dt = \frac{c}{mR} \frac{t^3}{3}$$

Now, what if  $\alpha = b\omega$ ? You will recognize a similar problem to this previously.

$$\alpha \equiv \frac{d\omega}{dt} \Rightarrow b\omega = \frac{d\omega}{dt} \Rightarrow b dt = \frac{d\omega}{\omega} \Rightarrow \int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = b \int_0^t dt \Rightarrow \ln\left(\frac{\omega}{\omega_0}\right) = bt \Rightarrow \omega = \omega_0 e^{bt}$$

Of course, here  $\omega_0$  can't be 0 since the particle won't start rotating at all, then.

Now what if  $\alpha = -h\theta$  where  $h$  is some constant.

Then:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = -h\theta \Rightarrow \frac{d^2\theta}{dt^2} + h\theta = 0$$

There are two cases to consider for  $h$ :

(1) let  $h$  be positive. The solutions for the angle are then:

$$\theta(t) = \theta_0 \cos(\omega_0 t + \phi); \omega_0 = \sqrt{h}$$

(this motion is known as simple harmonic oscillation which we'll study later)

(2) let  $h$  be negative. The solutions for the angle are then:

$$\theta(t) = \theta_0 e^{\pm\sqrt{h}t}$$

which is either exponential increasing or exponentially decreasing with time. Finally, the easiest of all situations is the case of a constant angular acceleration. In this case, you have:

$$\vec{\Gamma} = \vec{R} \times \vec{F} = \text{constant}$$

You can still apply the equations of motion to obtain instantaneous quantities:

$$\vec{\Gamma} = \frac{d\vec{L}}{dt} = I \cdot \frac{d\vec{\omega}}{dt}$$

(here, I've used more elaborate notation hinting at the tensor nature of  $I$  and the vector nature of  $\omega$ ). We are not going to that degree of depth here so we will look at rotations about one axis of the moment of inertia. Thus, dealing only with magnitudes, we have:

$$\gamma = I \frac{d\omega}{dt} = I\alpha \Rightarrow \alpha = \frac{\Gamma}{I} \Rightarrow \omega = \omega_0 + \left[\frac{\Gamma}{I}\right]t \Rightarrow \theta = \theta_0 + \omega_0 t + \frac{1}{2} \left[\frac{\Gamma}{I}\right]t^2$$

Again, you would also talk about average angular velocity if you so desire. This would be defined by:

$$\langle \omega \rangle = \frac{\int_0^t \omega(t) dt}{\int_{t=0}^t dt} = \omega_0 + \frac{\int_0^t \omega dt}{t} = \omega_0 + \frac{1}{2} \alpha t$$

And, the angle turned through would be given by:

$$\Delta \theta = \langle \omega \rangle t = \omega_0 t + \frac{1}{2} \alpha t^2$$

Notice that we could talk about average angular velocity in the same way as we did for translational quantities. Looking back at worksheet 2, we had:

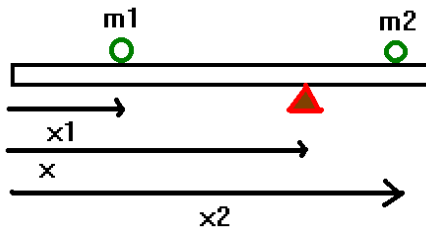
$$\langle v \rangle = v_0 + \frac{1}{2} \alpha t \quad .$$

Now if we multiply these quantities by  $r$  (assuming rigid body rotation pointing to a single mass) we have:

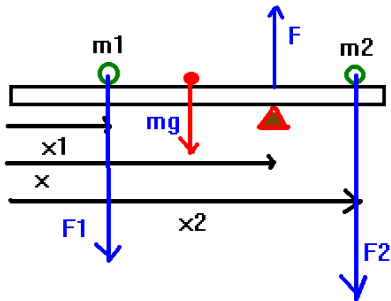
$$\langle v \rangle = \langle \omega \rangle r \Rightarrow \frac{v_0}{r} + \frac{1}{2} \frac{a}{r} t = \omega_0 + \frac{1}{2} \alpha t \Rightarrow \langle \omega \rangle = \omega_0 + \frac{1}{2} \alpha t \Rightarrow \Delta \theta = \langle \omega \rangle t = \omega_0 t + \frac{1}{2} \alpha t^2$$

(this is the approach for the non-calculus class here).

(2) A meter stick ( $L=1\text{m}$ ) of mass  $m=0.05\text{ kg}$  is placed on a pivot at  $x=0.7\text{ m}$ . A mass  $m_1=0.5\text{ kg}$  is placed at  $x_1=0.3\text{ m}$ . Where ( $x_2$ ) would a mass  $m_2=0.8\text{ kg}$  need to be placed in order to balance the system?



Solution: You will want to sketch in the forces present in this problem. Thus:



The condition for static equilibrium is:  $\sum \vec{F} = \vec{0}$ :  $\sum_{\text{any axis}} \vec{\Gamma} = \vec{0}$  .

Let's apply the first condition:

$$\sum \vec{F} = 0 \Rightarrow F - m_1 g - mg - m_2 g = 0 \Rightarrow F = (m_1 + m + m_2)g$$

Choose the pivot point as our axis since one torque vanishes there. However, remember that the weight of the meter stick acts at its mass which is  $\frac{1}{2}$  of the distance from one end to the other end of the meter stick. I've also shown this (in red) in the second sketch.

Let's apply the second condition:

$$\sum \vec{\Gamma}_{\text{pivot}} = \vec{0} \Rightarrow +(m_1 g)[x - x_1] + (mg) \left[ x - \frac{L}{2} \right] + F[0] - (m_2 g)[x_2 - x] = 0$$

In this case, using the first condition is not necessary since I've chosen my axis carefully. You will also notice that I've kept the distances positive and correctly reflected the sign convention in the torques that if the rotation is counter-clockwise, the torque is positive.

Solving the second condition gives:

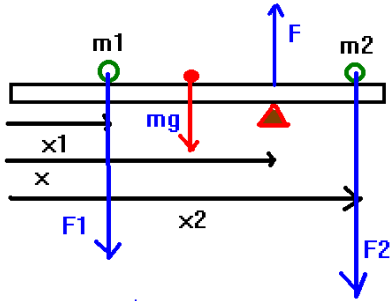
$$\begin{aligned}
 0 &= +(m_1 g)[x - x_1] + (mg)\left[x - \frac{L}{2}\right] + F[0] - (m_2 g)[x_2 - x] \\
 &\Rightarrow m_1[x - x_2]g + m\left[x - \frac{L}{2}\right]g = m_2 g[x_2 - x] \\
 &\Rightarrow m_1[x - x_1] + m\left[x - \frac{L}{2}\right] + m_2 x = m_2 x \\
 &\Rightarrow x_2 = \frac{m_1}{m_2}[x - x_1] + \frac{m}{m_2}\left[x - \frac{L}{2}\right] + x
 \end{aligned}$$

We can now put numbers on this result:

$$\begin{aligned}
 x_2 &= \frac{m_1}{m_2}[x - x_1] + \frac{m}{m_2}\left[x - \frac{L}{2}\right] + x = \frac{.5}{.8}[.7 - .3] + \frac{0.05}{.8} [.70 - .5] + .7 \\
 x_2 &= \frac{.5 \times .4}{.8} + \frac{.05 \times .2}{.8} + \frac{.7 \times .8}{.8} = \frac{.2 + .01 + .56}{.8} = \frac{.77}{.8} = 0.9625 \text{ m}
 \end{aligned}$$

(3) Solve problem 2 considering the axis to be at the center of mass (Not! The pivot).

A sketch of the situation remains the same as in problem 2:



The condition for static equilibrium is:  $\sum \vec{F} = \vec{0}$ :  $\sum_{\text{any axis}} \vec{\Gamma} = \vec{0}$

Let's apply the first condition:

$$\sum \vec{F} = \vec{0} \Rightarrow F - m_1 g - mg - m_2 g = 0 \Rightarrow F = (m_1 + m + m_2)g$$

Now, choose the center of mass as our axis.

Let's apply the second condition:

$$\sum \vec{\Gamma}_{\text{pivot}} = \vec{0} \Rightarrow m_1 g \left[ \frac{L}{2} - x_1 \right] + mg[0] + F \left[ x - \frac{L}{2} \right] - m_2 g \left[ x_2 - \frac{L}{2} \right] = 0$$

You will notice that I've kept the distances positive and correctly reflected the sign convention in the torques that if the rotation is counter-clockwise, the torque is positive. Solving the second condition gives:

$$\begin{aligned} m_1 \left[ \frac{L}{2} - x_1 \right] + m[0] + \frac{F}{g} \left[ x - \frac{L}{2} \right] &= m_2 \left[ x_2 - \frac{L}{2} \right] \\ \frac{m_1}{m_2} \left[ \frac{L}{2} - x_1 \right] + \frac{F}{m_2 g} \left[ x - \frac{L}{2} \right] &= \left[ x_2 - \frac{L}{2} \right] \\ \Rightarrow x_2 &= \frac{L}{2} + \frac{m_1}{m_2} \left[ \frac{L}{2} - x_1 \right] + \frac{F}{m_2 g} \left[ x - \frac{L}{2} \right] \end{aligned}$$

It is pretty clear that we're going to have to use the condition of F to complete this problem. Thus:

$$\begin{aligned} \Rightarrow x_2 &= \frac{L}{2} + \frac{m_1}{m_2} \left[ \frac{L}{2} - x_1 \right] + \frac{(m_1 + m + m_2)}{m_2} \left[ x - \frac{L}{2} \right] \\ x_2 &= \frac{L}{2} \left[ 1 + \frac{m_1}{m_2} - \frac{(m_1 + m + m_2)}{m_2} \right] + x_1 \left[ \frac{-m_1}{m_2} \right] + x \left[ \frac{(m_1 + m + m_2)}{m_2} \right] \\ \Rightarrow x_2 &= \frac{L}{2} \left[ \frac{m_2 + m_1 - m_1 - m_2 - m}{m_2} \right] + x_1 \left[ \frac{-m_1}{m_2} \right] + x \left[ \frac{(m_1 + m + m_2)}{m} \right] \\ \Rightarrow x_2 &= \left[ \frac{-m \frac{L}{2} - m_1 x_1 + (m_1 + m + m_2) x}{m_2} \right] \end{aligned}$$

let's put numbers on this now:

$$\begin{aligned} x_2 &= \left[ \frac{-m \frac{L}{2} - m_1 x_1 + (m_1 + m + m_2) x}{m_2} \right] = \frac{-0.05(0.5) - 0.5(.3) + (0.5 + 0.05 + 0.8)0.7}{0.8} \\ &= \frac{0.025 - 0.15 + 0.945}{0.8} = \frac{0.77}{.8} = 0.9625 \text{ m} \end{aligned}$$

The answers are the same.

(4) Show that a meter stick which is in equilibrium about one axis is in equilibrium about all axes.

Let's assume that the axis the meter stick is in equilibrium about is the origin. Then each force  $\vec{F}_i$  is acting at a position  $\vec{R}_i$  and the torque about the origin is given by:

$$\vec{\Gamma} = \sum \vec{\tau}_i = \sum \vec{R}_i \times \vec{F}_i = \vec{0}$$

Now, at some other axis  $\vec{R}_1$  the torques would be given by:

$$\vec{\Gamma}_1 = \sum [\vec{R}_1 - \vec{R}_i] \times \vec{F}_i$$

where the vector pointing from  $\vec{R}_i$  to  $\vec{R}_1$  is given by  $\vec{R}_1 - \vec{R}_i$ .

Thus, the torque about axis 1 is given by:

$$\vec{\Gamma}_1 = \sum [\vec{R}_1 - \vec{R}_i] \times \vec{F}_i = \sum \vec{R}_1 \times \vec{F}_i - \sum \vec{R}_i \times \vec{F}_i = \vec{R}_1 \times \sum \vec{F}_i - \sum \vec{R}_i \times \vec{F}_i$$

The last term in this expression is zero by the second condition for static equilibrium. Also, since the meter stick is in equilibrium, we have the first condition satisfied:

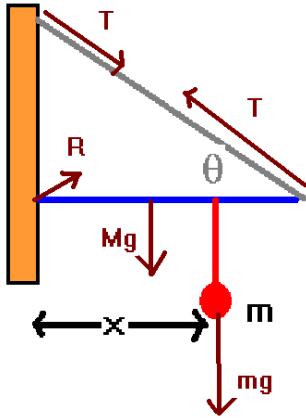
$$\sum \vec{F} = \vec{0}$$

Thus, we necessarily have:

$$\vec{\Gamma}_1 = \vec{R}_1 \times \sum \vec{F}_i - \sum \vec{R}_i \times \vec{F}_i = \vec{0} + \vec{0} = \vec{0}$$

Thus if the system is in static equilibrium about one axis, it is in static equilibrium about any axis.

(5) A rod of mass  $M$  and length  $L$  is attached to a wall as shown. A mass  $m$  is placed on the rod at a distance  $x$  from the wall. Find the tension in the cable, and the forces on the wall. Provide numerical values for the case  $L=1\text{m}$ ,  $M=2\text{kg}$ ,  $m=1\text{ kg}$ ,  $x=(1/3)\text{ m}$ ,  $\theta=45^\circ$ .



The system is in static equilibrium. Thus:  $\sum \vec{F} = \vec{0}$ :  $\sum_{\text{any axis}} \vec{\tau} = \vec{0}$  .

We need to choose an axis to calculate torques about.  
I will choose the point where the rod touches the wall.

The first condition gives:  $R_x - T_x = 0$ ;  $R_y - Mg - mg + T_y = 0$  .

We have from the second condition:  $-Mg\left(\frac{L}{2}\right) - mgx + T_y L = 0 \Rightarrow T_y = \frac{1}{2}Mg + mg\frac{x}{L}$

But this tension is related to the tension in the cable by:  $T_y = T \sin(\theta)$

$$T \sin(\theta) = \frac{1}{2}Mg + mg\frac{x}{L} \Rightarrow T = \frac{g}{\sin(\theta)} \left[ \frac{M}{2} + m\frac{x}{L} \right]$$

We can now find the reaction forces:

Having found  $T$ , we have  $T_x = T \cos(\theta) = R_x$  .

Having found  $T_y$ , we have:  $R_y = Mg + mg - T_y$

The angle that this force points at is:

$$\tan(\phi) = \frac{R_y}{R_x}$$

Numerical results for this problem:

$$T_y = \frac{1}{2}Mg + mg \frac{x}{L} = 9.8 + 9.8 \left(\frac{1}{3}\right) = 13.07 \text{ N}$$

$$T_y = T \sin(\theta) \Rightarrow T = \frac{T_y}{\sin(45)} = \frac{13.07 \text{ N}}{.707} = 18.48 \text{ N}$$

$$T_x = T \cos(\theta) = 13.07 \text{ N} = R_x$$

$$R_y = 29.4 \text{ N} - 13.07 \text{ N} = 16.33 \text{ N}$$

The angle is given by:  $\tan(\phi) = \frac{R_y}{R_x} = \frac{16.33}{13.07} = 1.249 \Rightarrow \phi = 51.33^\circ$

Among other things, this tells you how strong both your wall and your cable need to be to support the mass. Also you can see how strong the rod needs to be here. For example, you will need a minimum of 18.5N test fishing line to hold up this system. However, in building design, you're going to want to give significantly more safety than this, perhaps a factor of 10 or so at least. This means you'd choose 185N test fishing line in actuality.