Lorentz Transformations

How do they come about and what can you do with them?

A derivation of sorts

What we want to answer here is the following question: how do you transform from one set of coordinates to another set of coordinates when the two frames of reference have relative motion.

These are the (special) relativity transformations that relate moving frames to fixed frames.

Assume in system K a flashbulb goes off at t=t'=0, and the systems are coincident at this time. We will permit K' to move only along the x axis relative to K. Since the speed of light is the same in both systems, a spherical wavefront will be observed in both systems described by:

$$x^2+y^2+z^2=c^2t^2 \text{ and } x'^2+y'^2+z'^2=c^2t'^2$$

Now, think abut what lead to this initial conclusion: Both of these equations describe a sphere. Since the speed of light is the same in both frames, everyone in non-accelerated reference frames will observe a sphere. That is why both equations look the same.

These equations are inconsistent with Galilean relativity (not everyone in Galilean relativity sees the same speed for the speed of light).

Now here the relative motion is only going to be along the x-axis. We can pretty much always rotate to make this true in non-accelerated systems.

The y and z coordinates transform directly:

$$y'=y$$
$$z'=z$$

The simplest linear transformation which correctly relates velocity, position and time is:

$$x'=\gamma(x-vt) \text{ (x,t measured by an observer in K)}$$
$$x=\gamma(x'+vt') \text{ (x',t' measured by an observer in K')}$$

Perhaps some words are useful here: According to K, the observed x, v, and t must be able to predict x'. The required measurements are provided by K. The velocity is represented as – since this would imply a particle traveling in the +x direction. In general physics, we argued a similar concept with waves. The gamma here is as of yet pretty much an undetermined constant. Finally we want a simple yet general linear transformation since this is really the starting point for physics. Non-linear effects would be considered to be higher order corrections to a linear theory. However as I have previously pointed out, there does exist a derivation of these transformations directly.

What we are doing is to determine the value of γ.

<this is where the constancy of the speed of light comes in>
Both systems observe the light wave propagate at the speed of light.

This allows us to say: x=ct and x’=ct’. Thus:

$$ct' = \gamma(x - vt)$$
$$ct = \gamma(x' + vt')$$

we divide by c:

$$ct' = \gamma(ct - vt) \Rightarrow t' = \gamma t\left(1 - \frac{v}{c}\right)$$
$$ct = \gamma(ct' + vt') \Rightarrow t = \gamma t'\left(1 + \frac{v}{c}\right)$$

Now we are going to feed one of these equations into the other, substituting for t:
(I’m going to feed the second equation to the first equation)

$$t' = \gamma\gamma t'\left(1 + \frac{v}{c}\right)\left(1 - \frac{v}{c}\right) \Rightarrow 1 = \gamma^2\left(1 - \frac{v^2}{c^2}\right) \Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This is the requirement placed upon that previously undetermined quantity if we are to permit a linear transformation as indicated above.

Now that we know what gamma would have to be, let me show you how to transform time.

We have:

$$x' = \gamma(x - vt)$$
$$x = \gamma(x' + vt')$$

We already know what gamma is. To find out how time transforms, I’m going to feed the first equation to the second equation and solve it for time:

$$x = \gamma(\gamma(x - vt) + vt') = \gamma^2(x - vt) + \gamma vt'$$
$$\Rightarrow t' = \frac{1}{\gamma v}\left[x - \gamma^2(x - vt)\right] = \frac{x}{\gamma v}(1 - \gamma^2) + \gamma t$$

This tells us how t’ then depends upon t and x. The above equation tells us how t' depends upon x and t. We thus have the transformation noting that if x=0, $t' = \gamma t$.

On page 31, it is said that “a little algebra ..” let me show this to you:

$$\frac{1-\gamma^2}{\gamma v}=-\frac{\gamma\beta^2}{v}\left[\frac{v c^2}{\gamma v^2}\frac{1-\gamma^2}{\gamma v}\right]=-\frac{\gamma\beta^2}{v}\left[-\frac{c^2}{\gamma v^2}\right](1-\gamma^2)=-\frac{\gamma\beta^2}{v}\left[\frac{c^2}{\gamma^2 v^2}(\gamma^2-1)\right]=-\frac{\gamma^2\beta^2}{v}\left(1-\frac{1}{\gamma^2}\right]$$

$$\Rightarrow\frac{1-\gamma^2}{\gamma v}=-\frac{\gamma\beta^2}{v}\left[\frac{v c^2}{\gamma v^2}\frac{1-\gamma^2}{\gamma v}\right]-\frac{\gamma^2\beta^2}{v}\left[1-1+\frac{v^2}{c^2}\right]=-\frac{\gamma\beta^2}{v}$$

With that little bit of algebra, we then have the time transformation. This is also a portion of problem 10 at the end of the chapter.

This means that we can write the time transformation as:

$$t'=-\frac{\gamma\beta^2 x}{v}+\gamma t=\gamma\left[t-\frac{vx}{c^2}\right]=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

We are thus able to write the complete Lorentz transformations:

$$x'=\gamma(x-vt)$$
$$y'=y$$
$$z'=z$$
$$t'=\left[\frac{t-\frac{vx}{c^2}}{\sqrt{1-\beta^2}}\right]$$

The inverse transformation equations are obtained by replacing v by –v and exchanging the primes:

$$x=\gamma(x'+vt')$$
$$y=y'$$
$$z=z'$$
$$t=\left[\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\beta^2}}\right]$$

This, in principle, is it. All of special relativity is basically contained in these equations. The rest is really a matter of asking the right questions and keeping track of who is measuring what quantity.

I want to show you how time dilation comes about now before we go further.

Your text does this example on page 33 in the context of Mary, Frank and Melinda. Our conditions are these: K is fixed and K' is moving along x with v. Both Mary and Melinda are in K' with Melinda at $x_2^{'}$ and Mary at $x_1^{'}$. Frank is in K and lights a sparkler at a time t_1. The sparkler goes out at a time t_2. The **proper time** is the difference between these two times (i.e. the time difference that is observed by Frank in the K frame):

$$T_0 = t_2 - t_1$$

The 0 subscript denotes proper time.

Mary is beside the sparkler when it is lit. Melinda is beside the sparkler when it goes out.

One other provision: The clocks in both systems were synchronized at the instant Mary is beside the sparkler: this means that at this instant, Frank's clock, Mary's clock and Melinda's clock all read zero (and Melinda's clock synchronization is corrected for the finite speed of propagation of light). This could be mechanically imagined having clocks that are sensitive to light.

In the primed system, the report is delivered about the life time of the sparkler:
Mary was beside it and said it was lit at $t_1^{'}$
Melinda was beside it and said it went out at $t_2^{'}$
We subtract these times to get the lifetime of the sparkler:

$$T' = t_2^{'} - t_1^{'}$$

We can use the Lorentz transformations to relate T_0 to T':

$$T' = t_2^{'} - t_1^{'} = \frac{t_2 - \frac{v x_2}{c^2}}{\sqrt{1-\beta^2}} - \frac{t_1 - \frac{v x_1}{c^2}}{\sqrt{1-\beta^2}} = \frac{(t_2 - t_1) - \left(\frac{v}{c^2}\right)(x_2 - x_1)}{\sqrt{1-\beta^2}}$$

Now Frank stayed at the same position. This means that $x_1 = x_2$. The result then becomes:

$$T' = \frac{T_0}{\sqrt{1-\beta^2}}$$

This is the rigorous derivation of the time dilation that we have already talked about in class. It has enormous consequences.

Length Contraction

Observers in K and K' each have a meter stick at rest in their respective systems. The left end is at x_L' or x_L. The right end is at x_r or x_r'. Each observer measures the meter stick and reports a length given by the difference in coordinates. **In the rest frame of the meter stick, this length is called the proper length.** Now let K' move with a speed v along the x-axis. Let an observer in K measure the meter stick in K' so that $t=t_l=t_r$: the ends are measured simultaneously. Let's look at the results from the Lorentz transformations:

$$x_L^{'}=\gamma\left(x_L-v\,t_L\right) \text{ and } x_r^{'}=\gamma\left(x_r-v\,t_r\right)$$

The length difference is then:

$$x_r^{'}-x_L^{'}=\frac{\left(x_r-x_L\right)-v\left(t_r-t_L\right)}{\sqrt{1-\beta^2}}\Rightarrow L_0^{'}=\gamma L$$

Let me explain what this is: This is how much a meter stick existing and measured in the ' or moving frame would be measured by an observer in the fixed frame.

But since the proper lengths are all the same in **each** coordinate frame, we have:

$$L_0=\gamma L\Rightarrow L=\sqrt{1-\beta^2}\,L_0$$

The length measured by the fixed observer is seen to be shorter than the proper length (again the proper length being here that the meter stick is in the moving frame). This has important consequences for the shapes of objects according to different observers. What is a cube in one system may not be a cube in another system. It is, I believe, important to remember the important rules here that is Mary says Frank's meter stick is too short. So Mary's meter stick must be longer than Frank's meter stick.
For more details, look at page 38 in your text.

Relativistic Velocity addition

I like the example in your text on page 39 which discusses this problem:

Mary the space ship commander is traveling with a velocity v=0.6c. This is K'. Mary fires high speed protons which travel at 0.99c (according to Mary) in order to destroy asteroids. Frank in a space station (this is K) measures the speed of the protons. What does he measure?

Here, let u be the speed measured by Frank and u' be the speed measured by Mary.

We are going to use the Lorentz transformations to answer this. But first, you need to know which one you want to use. Look at the question. We know the speed of the protons in the primed frame and we want to know the speed in the unprimed frame. That suggests to start with this version of the Lorentz transformations:

$$x=\gamma(x'+vt')$$
$$y=y'$$
$$z=z'$$
$$t=\left[\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\beta^2}}\right]$$

We want, however, to find the speed. To do this, take the differentials:

$$dx=\gamma(dx'+v\,dt')$$
$$dy=dy'$$
$$dz=dz'$$
$$dt=\gamma\left(dt'+\frac{v}{c^2}dx'\right)$$

To obtain the velocities, we divide the differentials:

$$u_x=\frac{dx}{dt}=\frac{\gamma(dx'+v\,dt')}{\gamma\left(dt'+\frac{v}{c^2}dx'\right)}=\frac{\frac{dx'}{dt'}+v}{1+\frac{v}{c^2}\frac{dx'}{dt'}}=\frac{u_x'+v}{1+\frac{u_x'v}{c^2}}$$

So what did Frank measure?

$$u_x=\frac{u_x'+v}{1+\frac{u_x'v}{c^2}}=\frac{0.6\,c+0.99\,c}{1+(0.6)(0.99)}=\frac{1.590}{1.594}=0.997\,c$$

Now you can see that the speed is indeed greater than either v or u' but it is not in excess of the speed of light as would have been predicted by classical relativity.

We could go further than this to discuss what happens to accelerations. I like to stay away from that in this class in part because special relativity needs modifications when dealing with accelerations.

Now what if Mary fired the protons at Frank?

$$u_x=\frac{u_x'+v}{1+\frac{u_x' v}{c^2}}=\frac{-0.99\,c+0.6\,c}{1-(0.6)(0.99)}=\frac{-0.39\,c}{0.406}=-0.9606\,c$$

Well now that we have that, what about this: suppose Mary fired along her xy-axis. Then what does Frank measure? Note that u_x transforms as we have already shown. Again use these Lorentz transformations:

$$x=\gamma(x'+vt')$$
$$y=y'$$
$$z=z'$$
$$t=\left[\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\beta^2}}\right]$$

With the differentials:

$$dx=\gamma(dx'+v\,dt')$$
$$dy=dy'$$
$$dz=dz'$$
$$dt=\gamma\left(dt'+\frac{v}{c^2}dx'\right)$$

Then we have:

$$u_y=\frac{dy}{dt}=\frac{dy'}{\gamma\left(dt'+\frac{v}{c^2}dx'\right)}=\frac{dy'/dt'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}=\frac{u_y'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}$$

Also it is easy to see:

$$u_z=\frac{dz}{dt}=\frac{dz'}{\gamma\left(dt'+\frac{v}{c^2}dx'\right)}=\frac{dz'/dt'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}=\frac{u_z'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}$$

So, because of the time dilation, velocities along perpendicular directions here will show dilation which is in turn directly related to motion along x. However, suppose Mary fired only along y or z so that there was no x-component at all. Then the result is a bit simpler:

$$u_y=\frac{u_y'}{\gamma}:u_z=\frac{u_z'}{\gamma}$$

since $u_x'=0$ in this special situation.