QM Worksheet: Application to the square well

1DSWE:

$$i\hbar \frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V\psi(x,t)$$

(1) Assume $\psi=T(t)\varphi(x)$. Separate the equation into two equations, a spatial part and a time dependant part. Let the separation constant be E.
The answers are:

$$\frac{1}{T}\frac{dT}{dt} = -i\frac{E}{\hbar} \Rightarrow T(t) = e^{-i\frac{E}{\hbar}t}$$

$$\frac{d^2\varphi}{dx^2} - \frac{2m}{\hbar^2}V\varphi = -\frac{2mE}{\hbar^2}\varphi$$

(2) Assume a well exists bounded by infinite potentials at $x<0$ and $x>L$. Inside the well, the potential is zero. Find solutions to φ.

(a) Show the 1DTISWE is $\frac{d^2\varphi}{dx^2} + \frac{2mE}{\hbar^2}\varphi = 0 \Rightarrow \frac{d^2\varphi}{dx^2} + k^2\varphi = 0$. (Identify k as $k^2 = \frac{2mE}{\hbar^2}$).

(b) Assume the solution for φ: $\varphi = A\sin(kx) + B\cos(kx)$. Evaluate the solution at x=0 to argue by B must be zero.

(c) Evaluate the solution at x=L to show why $k = \frac{n\pi}{L}; n = 1, 2, \ldots$

(d) Find the energy eigenvalues for the square well.

(d) Write the solution for φ_n in terms of what k is.

(e) Normalize φ_n which means solving for A. The answer is $A = \sqrt{\frac{2}{L}}$

(f) Find the time dependence of a particle in the nth eigenstate.

Answers: $\psi_n(x,t) = \sqrt{\frac{2}{L}}e^{-i\left[n^2\omega_1 t\right]}\sin\left(n\pi\frac{x}{L}\right); E_n = \hbar\omega_n = n^2E_1 \Rightarrow \omega_n = n^2\frac{E_1}{\hbar} = n^2\omega_1$

Measurements with the eigenfunctions

Properly normalized eigenfunctions for the particle: $\varphi_n = \sqrt{\frac{2}{L}}\sin\left(n\pi \frac{x}{L}\right)$

(a) Find <x>: Do this: $\left\langle \varphi_n \middle| \tilde{x} \middle| \varphi_n \right\rangle$. Answer: $\langle x \rangle = \frac{L}{2}$

(b) Find <x^2>: Do this: $\left\langle \varphi_n \middle| \tilde{x}^2 \middle| \varphi_n \right\rangle$. Answer: $\left\langle x^2 \right\rangle = L^2\left[\frac{1}{3} - \frac{1}{2n^2\pi^2}\right]$

(c) Find Δx. Do this: $\Delta x \equiv \sqrt{\left\langle x^2 \right\rangle - \langle x \rangle^2}$

(d) Find $\langle p \rangle$. Do this: $\langle p \rangle = \left\langle \varphi_n \middle| \tilde{p} \middle| \varphi_n \right\rangle = -i\hbar \left\langle \varphi_n \middle| \frac{d}{dx} \middle| \varphi_n \right\rangle = 0$ (why?) (2 reasons)

(e) Find <p^2>. Do this: $\left\langle p^2 \right\rangle = 2m\langle E \rangle = 2m\left\langle \varphi_n \middle| \tilde{H} \middle| \varphi_n \right\rangle = 2mn^2E_1$ (Why?)

(f) Find Δp. Do this: $\Delta p = \sqrt{\left\langle p^2 \right\rangle - \langle p \rangle^2}$.

(g) Find the uncertainty relation for the nth eigenstate.
Answer: $(\Delta P)(\Delta X) = [n\frac{\hbar\pi}{L}]\left[\frac{L}{\sqrt{24}n\pi}\sqrt{2n^2\pi^2 - 12}\right] = \frac{\hbar}{\sqrt{24}}\sqrt{2n^2\pi^2 - 12}$

Non-Commutation of operators.

Consider two measurements of position and momentum. We want to do this first: $\langle px \rangle = \left\langle \varphi_n \middle| \tilde{p}\tilde{x} \middle| \varphi_n \right\rangle$. Next do this: $\langle xp \rangle = \left\langle \varphi_n \middle| \tilde{x}\tilde{p} \middle| \varphi_n \right\rangle$. Find the difference between these for the square well eigenfunctions. See if it is given by:

$$\left[\tilde{X},\tilde{P}\right] \equiv \tilde{X}\tilde{P} - \tilde{P}\tilde{X} = -i\hbar x \frac{\partial}{\partial x} + i\hbar + i\hbar x \frac{\partial}{\partial x} = i\hbar$$