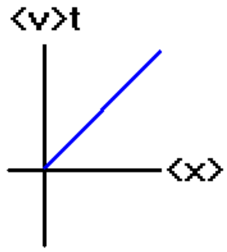
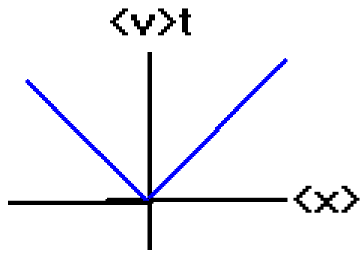
An simple introduction to space – time diagrams

Suppose that you can drive with an average velocity of <v>.
If you start from the origin at t=0, what is the average distance you travel in a time t?
The answer is, of course: $\langle x \rangle = \langle v \rangle t$.
Let's now plot a graph of x vs <v>t: The slope of this graph will be 1.

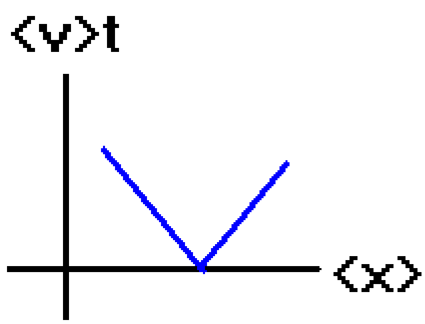


Now suppose your car can drive in the + direction and also in the – direction. A graph of this would look like the following:



In a way, for your car this represents only the two possible trajectories that it can take.

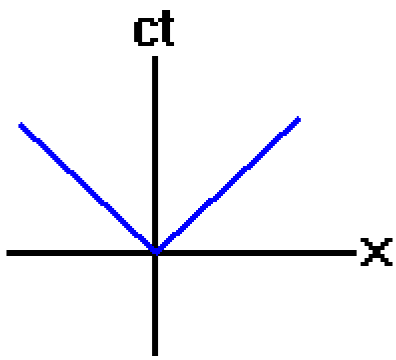
Next, suppose that you are not at the origin at t=0. A plot of this would appear as follows:



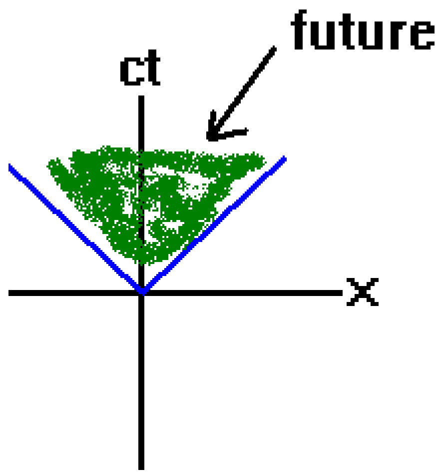
Next, let's assume that you have not actually started to drive yet. Then this graph only represents possible positions at particular times. The intersection of these two lines at t=0 would always be at the present time. The two lines simply represent your future positions.

Since you can only drive as I have specified, it is not possible for your space-time positions to be anywhere except on the blue line. Now let's go just a little bit further.

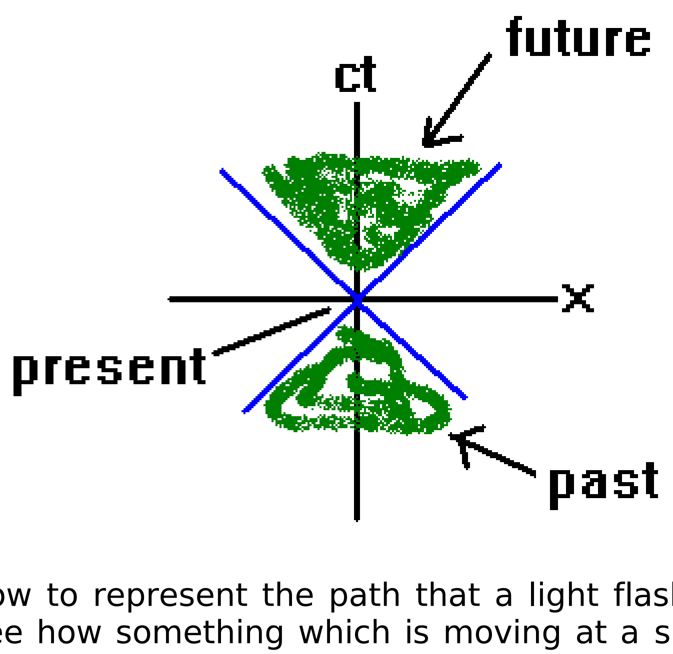
What your car represents is, in fact, a pulse of light which can travel at only c, the speed of light. In this thought, we redraw the space-time diagram to show that:



From this diagram, it is pretty clear where the future is. I have colored this region.

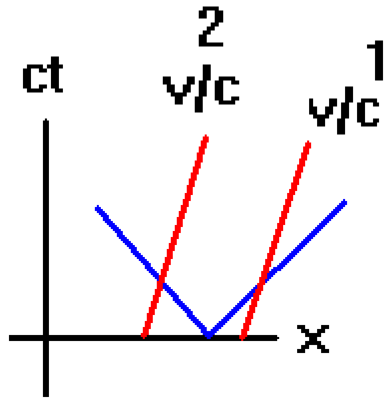


The present is the intersection. The past in fact then is this cone but upside down. Everything but light could have been in the green region shown.



Now that we know how to represent the path that a light flash will take on a space-time diagram, let's see how something which is moving at a speed less than c would be represented. Consider this problem: In Frank's reference frame, an event happens (perhaps the flash of a bulb) at some location along the x-axis.

Mary and George (Moving and Going) are both traveling with a velocity v in the +x direction but are separated by a distance L_0^{MG} as observed in either the M frame or the G frame. Although this distance is moving relative to F it is still a proper length because it is at rest in the M or G frames.



The intersections of these curves represents the times at which the pulses are received by 1 and 2 respectively. What the red curves represent are the motions of 1 and 2 (Mary in frame K' and George) as observed by the fixed observer (Frank, in frame K).

Since the speed of light is the same in all frames of reference, each observer will potentially see a spherical shell of light leaving the flash. The equation for a sphere of radius R is given by:

$$R^2 = x^2 + y^2 + z^2$$

The radius of this sphere is the distance that the light pulse has traveled in a time t. Thus:

$$c^2 t^2 = x^2 + y^2 + z^2$$

What I want to prove now is that there is a quantity that, under the Lorentz transformations, is invariant in all reference frames. Consider:

$$\text{in } K: s^2 = x^2 - (ct)^2$$

I might say, what this represents is one of the following cases:

(a) Inside the sphere ($s^2 < 0$)
(b) On the sphere ($s^2 = 0$)
(c) Outside the sphere ($s^2 > 0$)

We want to see how this appears in the primed frame (K'). We can answer this easily enough by using the transformations I've shown earlier:

$$x = \gamma(x' + vt') : y = y' : z = z' : t = \frac{t + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$x' = \gamma(x - vt) : y' = y : z' = z : t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

To do this, we substitute in for x and t and then do algebra:

$$(s')^2 = \gamma^2 (x' + vt')^2 - c^2 \frac{\left(t' + \frac{vx'}{c^2}\right)}{(1-\beta)^2}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

Here is the algebra:

$$(s')^2 = \gamma^2 (x' + vt')^2 - \gamma^2 c^2 \left[\left(t' + \frac{vx'}{c^2}\right)^2\right]$$

$$(s')^2 = \gamma^2 (x' + vt')^2 - \gamma^2 \left[(ct' + \beta x')^2\right]$$

$$(s')^2 = \frac{1}{1-\beta^2}(x' + vt')^2 - \frac{1}{1-\beta^2}\left[(ct' + \beta x')^2\right]$$

$$(s')^2 = \left[x'^2 + 2vx't' + v^2 t'^2\right]\frac{1}{1-\beta^2} - \frac{1}{1-\beta^2}\left[c^2 t'^2 + 2\beta x't' + \beta^2 x'^2\right]$$

$$(s')^2 = x'^2\left[\frac{1}{1-\beta^2}\right]\left[1-\beta^2\right] - (c^2 t'^2)\left[\frac{1}{1-\beta^2}\right]\left[1-\beta^2\right] + \left[\frac{1}{1-\beta^2}\right]\left[2vx't' - 2\beta c x't'\right]$$

$$(s')^2 = x'^2\left[\frac{1}{1-\beta^2}\right]\left[1-\beta^2\right] - (c^2 t'^2)$$

$$\Rightarrow (s')^2 = x'^2 - (c^2 t'^2)$$

Now look at the forms: they are the same. This means that the quantities:

$$\text{in } K: s^2 = x^2 - (ct)^2$$

$$\text{in } K': s'^2 = x'^2 - (ct')^2$$

are the same in all reference frames. This is called an invariant for this reason.

In fact, it is the calculation of Δs that provides information about causality.
A calculation of this is given by the following example:

Consider 2 events which have the following space-time coordinates:

$(x_{1,} ct_1):(x_{2,} c\,t_2)$

Now it may be that event 1 caused event 2, or event 2 caused event 1. Both of these can not be the case, however it is possible that neither of these is the case. It can be calculated as follows:

(a) did event 1 cause event 2? (**The Minkowski Metric**)

$$(\Delta s)^2 = (\Delta x)^2 - c^2 (\Delta t)^2$$

There are the following possibilities here:

$(\Delta s)^2 = 0$: the two events could only be connected by a light signal (this is called a light-like separation). Event 1 could have caused event 2.

$(\Delta s)^2 > 0$: The two events can not be connected by any signal (this is called a space-like separation).Event 1 could not have caused event 2.

$(\Delta s)^2 < 0$: The two events can be connected by a signal (this is called a time-like separation). Event 1 could have caused event 2.

Let's take a few simple examples.

(a) A single light-machine thing is located at x. It flashed at $t_1=0$. Later, when $ct_2=1$, it flashed again. Could the first flash have caused the second? The obvious answer here is yes, but let's calculate it.

In this case, since the machine stays in the same place, the increment $(\Delta x)^2$ is zero. This shows then that $(\Delta s)^2$ is always negative. According to our test, event 1 could have caused event 2.

It is interesting, before we go further, to see if there exists a reference frame in which the two pulses could be simultaneous. The quick answer to this is no, and the reason is that as this is stated, the measurement of the increment between the two pulses is the proper time, and thus the smallest time increment possible. However, let's look at the transformations to show this also. The two important transformations here are:

$$x'=\gamma(x-vt):t'=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

Both of the events happen at x=0. So we have:

$$x_1'=-\gamma vt_1:t'=\frac{t}{\sqrt{1-\beta^2}}$$

$$x_2'=-\gamma vt_2:t'=\frac{t}{\sqrt{1-\beta^2}}$$

Event 1 happens at $t_1=0$. This further simplifies the transformations to read:

$$x_1'=0:t_1'=0$$

$$x_2'=0:t_2'=\frac{t_2}{\sqrt{1-\beta^2}}$$

The time increment between the two events is then given by:

$$\Delta t'=\frac{t_2}{\sqrt{1-\beta^2}}$$

The smallest non-negative value of this time increment will be when t_2 is also zero. But if t_2 is zero, the two events are simultaneous in the rest frame. In fact, in this special case, the events will be simultaneous in all rest frames since essentially they may be regarded as a single event.

(b) The first light-machine thing is located at $x_1=0$ and flashes at $t_1=0$. A Second light machine thing is located at $x_2=bct$ where t_2 is, in this case going to be a set amount of time. What is the maximum value of b so that the two events can be causally connected if machine 2 flashes at t_2 and machine 1 flashes at $t_1=0$?

The obvious answer here is the following: the separation in space between the two machines is given by:

$$\Delta x=x_2-x_1=bct_2$$

Since the fastest that any signal can travel between these two machines is the speed of light, we can calculate the required velocity for causality. It is:

$$\text{distance}=\text{rate x time}\Rightarrow \text{rate}=\frac{\text{distance}}{\text{time}}=\frac{bct_2}{t_2}=bc$$

This means that the maximum value for b is 1. In this event, (so to speak) the two events are light-like. Now let's get the same thing by direct calculation.

$$(\Delta x)^2=b^2c^2t_2^2:c^2(\Delta t)^2=c^2t_2^2\Rightarrow(\Delta s)^2=c^2t_2^2[b^2-1]$$

If b is greater than 1, the separation is space like and the two events can not be causally connected. If b=1, the separation is light like and the two events can be connected by a light signal. If b<1, the separation is time-like and the two events can be connected by a signal with a speed less than the speed of light. The maximum value for causality is b=1, but for values of b between 0 and 1, the events may also be causally connected.

It is interesting, before we go further, to see if there exists a reference frame in which the two pulses could be simultaneous. The two important transformations here are:

$$x'=\gamma(x-vt):t'=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

Event 1 happens at x=0, $t_1=0$. So $x_1'=0:t_1'=0$

Event 2 happens at $x_2=bct_2;t=t_2$. So:

$$x_2'=\gamma(bct_2-vt_2):t_2'=\frac{t_2-\frac{vbct_2}{c^2}}{\sqrt{1-\beta^2}}$$

It is clear from the second equation that for the events to be simultaneous,

$$t_2'=\frac{t_2-\frac{vbct_2}{c^2}}{\sqrt{1-\beta^2}}=0\Rightarrow t_2\left(1-b\frac{v}{c}\right)=0\Rightarrow b\frac{v}{c}=1\Rightarrow v=\frac{c}{b}$$

Assuming b=1, this means that the frame of reference in which this happens is moving at the speed of light. Thus there is no frame of reference in which these two events will be simultaneous if causally connected by only a light signal. This constant b, however, can have other values. If b>1, there will exist reference frames in which the two events happen simultaneously (but are not causally connected). Here is a very specific example: suppose b=2. Then the frame K' in which the events are simultaneous is one moving with ½ the speed of light.

Another example.

Two events have space-time coordinates given by:

$$\left(x_{1,}t_1\right)=\left(6x10^4\,m,2x10^{-5}\,s\right):\left(x_{2,}t_2\right)=\left(1x10^5\,m,1x10^{-5}\,s\right)$$

Find a reference frame(K') in which the two events are simultaneous and then find the spatial separation of the two events in K'.

Let's first determine if it is possible:

$$\Delta s^2=x^2-\left(ct\right)^2=\left(10x10^4-6x10^4\right)^2-c^2\left(1x10^{-5}-2\,x10^{-5}\right)^2=16x10^8-\left(9x10^{16}\right)\left(1x10^{-10}\right)>0$$

Since the Minkowski Metric is greater than zero, the separation is space-like and the two events can not be connected by a signal. The two events therefore can not be causal.

For this set of coordinates, there is no frame in which the desired result is possible.

The boundary between a possible solution and an impossible solution is:

$$\frac{\Delta x}{\Delta t}=c$$

If the magnitude of this space-time separation ratio (which is **not** a velocity) is less than the speed of light, the problem is impossible.

Here is the algebra behind this statement:

The Lorentz transformations are:

$$x_1^{'}=\gamma\left(x_1-vt_1\right):t_1^{'}=\frac{t_1-\frac{vx_1}{c^2}}{\sqrt{1-\beta^2}}$$

$$x_2^{'}=\gamma\left(x_2-vt_2\right):t_2^{'}=\frac{t_2-\frac{vx_1}{c^2}}{\sqrt{1-\beta^2}}$$

For simultaneity, there exists a reference from such that:

$$t_1^{'}=t_2^{'}\Rightarrow t_2^{'}-t_1^{'}=\frac{\left(t_2-t_1\right)-\frac{v}{c^2}\left(x_2-x_1\right)}{\sqrt{1-\beta^2}}=0$$

$$\Rightarrow\left(t_2-t_1\right)-\frac{v}{c^2}\left(x_2-x_1\right)=\left(t_2-t_1\right)=\frac{v}{c^2}\left(x_2-x_1\right)$$

$$\Rightarrow\frac{v}{c^2}=\frac{\left(t_2-t_1\right)}{\left(x_2-x_1\right)}\Rightarrow v=c^2\frac{\left(t_2-t_1\right)}{\left(x_2-x_1\right)}\Rightarrow\frac{v}{c}=c\frac{\left(t_2-t_1\right)}{\left(x_2-x_1\right)}$$

This, in fact, tells you the beta factor that the coordinate system must move at for two events to be simultaneous.

Consider this set of coordinates:

$$\left(x_{1,}\, t_1\right)=\left(6x10^4\, m, 2x10^{-5}\, s\right):\left(x_{2,}\, t_2\right)=\left(1x10^4 m, 1x10^{-5}\, s\right)$$

$$v=c^2\frac{\left(t_2-t_1\right)}{\left(x_2-x_1\right)}\Rightarrow\frac{v}{c}=c\frac{\left(t_2-t_1\right)}{\left(x_2-x_1\right)}=c\frac{1x10^{-4}-2x10^{-4}}{1x10^4-6\,x\,10^4}=c\frac{-1x10^{-4}}{-5\,x\,10^4}=0.6$$

This problem is therefore possible and this is the beta factor of the coordinate system that would be required for the events to be simultaneous. The spatial separation in this frame can also be determined:

$$x_1^{'}=\gamma\left(x_1-v t_1\right):x_2^{'}=\gamma\left(x_2-v\,t_2\right)\Rightarrow x_2^{'}-x_1^{'}=\gamma\left(x_2-x_1-v\left(t_2-t_1\right)\right)$$

Here,

$$\beta=0.6\Rightarrow\gamma=1.25$$

So:

$$\Delta x'=1.25\left(-5x10^5-1.8x10^8\left(-1x10^{-4}\right)\right)=8.5x10^4 m$$

I have produced an interactive space-time spreadsheet (ver 0.3) which you can vary the times and positions of events to see what the different viewers will observe.

Quick Retrace: Length Measurements

Now that you know how to work with events, let's see about length contraction.

It is important here to understand how to make a measurement of length in a moving reference frame. In fact, measurement of length of a moving object needs to be done in such a was so that the ends are measured simultaneously. If the object is measured in its rest frame, the end coordinates need not be measured simultaneously and this length is called the "Proper Length" (L_0) of the object.

Consider a ruler at rest in frame K'. The question is this: If an observer in K' measures the ruler, what would be the measurement of the same ruler in frame K? In other words, how do the length measurements between K and K' transform?

The connection between the two lengths is obtained by the Lorentz transformation:

$$x_1' = \gamma(x_1 - vt) : x_2' = \gamma(x_2 - vt)$$

Subtract these two results to give the transformation:

$$x_2' - x_1' \equiv L_0 = \gamma(x_2 - x_1) = \gamma L \Rightarrow L = \frac{1}{\gamma} L_0 = \sqrt{1-\beta^2} L_0$$

An observer in a fixed reference frame trying to measure a moving ruler would therefore measure a length which is shorter than what would be measured in the rest frame of the ruler. This is called length contraction.

The important thing to remember as a check is that objects in motion contract. Here are a few examples:

(1) How fast would a meter stick need to fly in order to be contracted by 50%, as measured by an observer at rest?

$$L = \sqrt{1-\beta^2} L_0 \Rightarrow \frac{L}{L_0} = 0.5 = \sqrt{1-\beta} \Rightarrow 0.25 = 1 - \beta^2 \Rightarrow \beta^2 = 0.75 \Rightarrow \beta = 0.866 \Rightarrow v = 0.866c$$

(2) Later in the course, you will solve the quantum mechanical square well problem. As a first step towards corrections to that problem, consider a square well which is of length $L_0 = 1\text{x}10^{-9}$ m in the rest frame of the lab. If a particle is trapped in this square well, and the speed of the particle is v=0.9c, what length does the particle measure for the square well?

$$L = \sqrt{1-\beta^2} L_0 \Rightarrow \frac{L}{L_0} = \sqrt{1-0.9^2} \Rightarrow \frac{L}{L_0} = 0.436 \Rightarrow L = 4.36\text{x}10^{-10}\,\text{m}$$

Based upon this viewpoint, I guess we can say that square wells aren't. However, there is a little bit more to the story than this.

(3) A cube measures 1m^3 in its rest frame. How fast would the cube need to be moving so that a fixed observer would see the volume of the cube to be reduced by ½?

The key point here is that only the side of the cube parallel to the motion is affected, so that the speed is, in fact, the same as in problem 1. If the motion is not parallel to an edge, the problem is a bit more complicated.

(4) Germany has some fairly high speed trains. Suppose a 1km train (in its rest frame) moved at a speed of 360 km/hr. How much shorter would a fixed observer measure the length of the train to be than a co-moving observer?

$$360\frac{\text{km}}{\text{hr}} \text{x} \frac{1\text{hr}}{3600\text{s}} \text{x} \frac{1000\text{m}}{1\text{km}} = 100\frac{\text{m}}{\text{s}} : \beta = \frac{v}{c} = \frac{100}{3\text{x}10^8} = 3.3\text{x}10^{-7}$$

The problem here is that it is hard to get a good approximation to the result. However, expand the quadratic in a series:

$$\left[1-\beta^2\right]^{1/2} \approx 1 - \frac{\beta^2}{2} \Rightarrow L_0 - L_0\frac{\beta^2}{2} = L \Rightarrow L_0 - L \approx L_0\frac{\beta^2}{2} = 500\left(3.3\text{x}10^{-7}\right)^2 = 5.445\text{x}10^{-11}\text{m}$$