Relativistic Acceleration Transformation

Consider Mary in a frame moving along the x axis with a velocity v relative to Frank. In the moving frame, a particle moving with a velocity $\vec{u}'$ has an acceleration $\vec{a}'$. What does Frank observe? Note here that it is not Mary's frame which is accelerating: we are only transforming her observations of an acceleration to Frank's frame.

The Lorentz transformations are:

$$x=\gamma(x'+vt'):y=y':z=z':t=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$x'=\gamma(x-vt):y'=y:z'=z:t'=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

Let me get the velocity transforms first.

$$x=\gamma(x'+vt'):y=y':z=z':t=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$\frac{dx}{dt}=u_x=\frac{\gamma(dx'+v\,dt')}{\left[\frac{dt'+\frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}\right]}=\frac{u_x'+v}{1+\frac{u_x'v}{c^2}}$$

$$\frac{dy}{dt}=u_y=\frac{dy'}{dt'}=\frac{dy'}{\left[\frac{dt'+\frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}\right]}=\sqrt{1-\beta^2}\,dy'\frac{\frac{dy'}{dt'}}{\left[1+\frac{v}{c^2}\frac{dx'}{dt'}\right]}=\sqrt{1-\beta^2}\frac{u_y'}{\left[1+\frac{v}{c^2}u_x'\right]}=\frac{\frac{u_y'}{\gamma}}{1+\frac{v}{c^2}u_x'}$$

$$\frac{dz}{dt}=u_z=\frac{dz'}{dt'}=\frac{dz'}{\left[\frac{dt'+\frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}\right]}=\sqrt{1-\beta^2}\,dz'\frac{\frac{dz'}{dt'}}{\left[1+\frac{v}{c^2}\frac{dx'}{dt'}\right]}=\sqrt{1-\beta^2}\frac{u_z'}{\left[1+\frac{v}{c^2}u_x'\right]}=\frac{\frac{u_z'}{\gamma}}{1+\frac{v}{c^2}u_x'}$$

Remember: we showed in class the symmetry of replacing the + with a - and the primes to obtain the inverse transformations.

Now to obtain the acceleration transformations, we take the differential increments again:

$$\frac{du_x}{dt} : \frac{du_y}{dt} : \frac{d u_z}{dt} \quad .$$

The resulting transformations are given by:

$$a_x = \frac{a_x'}{\gamma^3 \left[1 + \frac{u_x' v}{c^2}\right]^3} : a_y = \frac{1}{\gamma^2 \left[1 + \frac{u_x' v}{c^2}\right]^2} \left[a_y' - a_x' \frac{u_y'}{\left[1 + \frac{u_x' v}{c^2}\right]} \frac{v}{c^2} \right] : a_z = \frac{1}{\gamma^2 \left[1 + \frac{u_x' v}{c^2}\right]^2} \left[a_z' - a_x' \frac{u_z'}{\left[1 + \frac{u_x' v}{c^2}\right]} \frac{v}{c^2} \right]$$

The transformations for the accelerations are not nearly as clean-cut as for the velocities, but one important point is shown here: where-as in the classical mechanics, accelerations are orthogonal along the three Cartesian coordinates when transformed, under special relativity this is not the case.

If the particle is moving along the x-axis only or if the acceleration is zero in the x-direction, these transformations reduce to become:

$$a_x = a_x' : a_y = \frac{a_y'}{\gamma^2 \left[1 + \frac{u_x' v}{c^2}\right]^2} : a_z = \frac{a_z'}{\gamma^2 \left[1 + \frac{u_x' v}{c^2}\right]^2}$$

Note also this has pretty important implications for how forces would transform from Mary's frame to Frank's frame. As something to look at in particular, suppose in Mary's frame, a gravitational acceleration $\vec{g}' = -g\hat{y}'$ exists on a particle and Mary is traveling past Frank's frame in the x-direction with a speed βc. What does Frank say happens? (Note that to probably do this more correctly we should use the inverse acceleration transforms and let Frank observe the acceleration g.

The solution:

$$a_x = 0 : a_y = \frac{1}{\gamma^2 \left[1 + \frac{u_x' v}{c^2}\right]^2} \left[-g'\right] : a_z = 0$$

Which means that we also need to know the x-component of the particle's velocity. Furthermore, suppose Mary also observed that the particle had no velocity in the x direction. Then the acceleration becomes:

$$a_x = 0 : a_y = \frac{-g}{\gamma^2} : a_z = 0$$

Which means that we also need to know the x-component of the particle's velocity. If the particle is at rest with respect to Mary, then there is no problem. However if Mary sees the particle traveling backward with $u_x' = -\beta c$, then in Frank's frame we have:

$$a_x = 0 : a_y = \frac{1}{\gamma^2 \left[1 - \beta^2\right]^2} \left[-g'\right] = \frac{\gamma^4}{\gamma^2} \left[-g'\right] = -\gamma^2 g' : a_z = 0 \quad .$$

Note that the particle may have a velocity in the y' direction to give the same result.

Derivation of a_x:

$$u_x=\frac{u'_x+v}{1+\frac{u'_x v}{c^2}}:d\,u_x=\frac{d\,u'_x}{1+\frac{u'_x v}{c^2}}-\frac{u'_x+v}{\left(1+\frac{u'_x v}{c^2}\right)^2}\left(\frac{v}{c^2}\right)du'_x:dt=\gamma\,dt'+\gamma\frac{v}{c^2}dx'$$

$$\frac{du_x}{dt}=\frac{\frac{d\,u'_x}{1+\frac{u'_x v}{c^2}}-\frac{u'_x+v}{\left(1+\frac{u'_x v}{c^2}\right)^2}\left(\frac{v}{c^2}\right)du'_x}{\gamma\left(dt'+\frac{v}{c^2}dx'\right)}=\frac{a'_x}{\gamma}\left(\frac{\frac{1}{1+\frac{u'_x v}{c^2}}-\frac{u'_x+v}{\left(1+\frac{u'_x v}{c^2}\right)^2}\left(\frac{v}{c^2}\right)}{1+\frac{u'_x v}{c^2}}\right)$$

$$=\frac{a'_x}{\gamma}\frac{\left(1-\frac{u'_x+v}{\left(1+\frac{u'_x v}{c^2}\right)}\left(\frac{v}{c^2}\right)\right)}{\left(1+\frac{u'_x v}{c^2}\right)^2}=\frac{a'_x}{\gamma}\frac{\left(1+\frac{u'_x v}{c^2}-\left(u'_x+v\right)\left(\frac{v}{c^2}\right)\right)}{\left(1+\frac{u'_x v}{c^2}\right)^3}=\frac{a'_x}{\gamma^3}\frac{1}{\left(1+\frac{u'_x v}{c^2}\right)^3}$$

$$\text{if } u'_x=0\Rightarrow a_x=\frac{a'_x}{\gamma^3}$$

Derivation of a_y and a_z :

$$u_y=\frac{\frac{u'_y}{\gamma}}{1+\frac{v}{c^2}u'_x}:d\,u_y=\frac{1}{\gamma}\left(\frac{d\,u'_y}{1+\frac{u'_x v}{c^2}}-\frac{u'_y}{\left(1+\frac{u'_x v}{c^2}\right)^2}\left(\frac{v}{c^2}\right)du'_x\right):dt=\gamma\,dt'+\gamma\frac{v}{c^2}dx'$$

$$a_y=\frac{1}{\gamma^2}\frac{\left(\frac{a'_y}{1+\frac{u'_x v}{c^2}}-\frac{u'_y}{\left(1+\frac{u'_x v}{c^2}\right)^2}\left(\frac{v}{c^2}\right)a'_x\right)}{1+\frac{u'_x v}{c^2}}=\frac{1}{\gamma^2\left(1+\frac{u'_x v}{c^2}\right)^2}\left(a'_y-\frac{u'_y}{1+\frac{u'_x v}{c^2}}\frac{v}{c^2}a'_x\right)$$

$$\text{if } a'_x=0 \text{ or } u'_y=0\Rightarrow a_y=\frac{a'_y}{\gamma^2\left(1+\frac{u'_x v}{c^2}\right)^2}$$

$$a_z=\frac{1}{\gamma^2\left(1+\frac{u'_x v}{c^2}\right)^2}\left(a'_z-\frac{u'_z}{1+\frac{u'_x v}{c^2}}\frac{v}{c^2}a'_x\right):$$

$$\text{if } a'_x=0 \text{ or } u'_z=0\Rightarrow a_z=\frac{a'_z}{\gamma^2\left(1+\frac{u'_x v}{c^2}\right)^2}$$

Motion of a particle undergoing a constant acceleration.

This follows the work of Baglio and Bethermin from ENC Cachan France.

In order to discuss the motion of a particle under a constant acceleration, it is necessary to realize that the acceleration should be seen to be constant in the frame of reference of the particle. Consider a frame of reference which is moving along the +x direction but we permit the possibility that this frame may be undergoing an acceleration in the instantaneous rest from of the accelerating frame.

We have the transformation of the accelerations in this frame given by:

$$a_x = \frac{a_x^{'}}{\gamma^3 \left[1 + \frac{u_x^{'} v}{c^2}\right]^3}$$

Since in the prime frame there is no relative motion of the particle (since this is the instantaneous frame we have chosen), we then have:

$$a_x = \frac{a_x^{'}}{\gamma^3} \Rightarrow a_x^{'} = \gamma^3 a_x = \gamma^3 \frac{dv_x}{dt}$$

We require that a'_x is a constant in the particle's frame. We then have this as the differential equation of motion:

$$a_x^{'} = \gamma^3 \frac{dv_x}{dt} \Rightarrow \int_{t=0}^{t} a_x^{'} dt = \int_{v_x=0}^{v_x} \gamma^3 d\, v_x = \int d\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) \Rightarrow a_x^{'} t = \frac{v_x}{\sqrt{1 - \frac{v_x^2}{c^2}}}$$

where I have set the constant of integration to zero (starts with an initial velocity of zero at t=0). This is solved then for the velocity.

$$\left(a_x^{'} t\right)^2 - \left(a_x^{'} t\right)^2 \frac{v_x^2}{c^2} = v_x^2 \Rightarrow v_x^2 \left(1 + \frac{\left(a_x^{'} t\right)^2}{c^2}\right) = \left(a_x^{'} t\right)^2 \Rightarrow v_x = \frac{a_x^{'} t}{\sqrt{1 + \frac{\left(a_x^{'} t\right)^2}{c^2}}}$$

From this it can be then confirmed that for a constant acceleration, acting for an infinite amount of time, the velocity is limited by the speed of light. This same result will happen also by looking at the relativistic moment of a particle when subject to an impulse. It is also important to note that the acceleration has broken the symmetry of the problem and strictly said this is not a problem that except for an instant, can be cast into the form of a Mary-Frank problem.