Lorentz transformations

It is not essential to require that the watches and length read 0 at 0. The Lorentz transformations are the simplest linear transformations which agree with the earlier two experiments. These though are new physics and the validity of them is if they hold up in experiment. And I want to point out that it is possible to derive these based upon the concepts of length contraction and time dilation. It is quite a complicated derivation but possible.
The Galilean transformations for an object observed with x, y, z, t in a fixed frame and characterized by $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ in the moving frame are:

1. Galilean transformations: $\begin{matrix}\bar{x}=x-vt\\ \bar{y}=y\\ \bar{z}=z\\ \bar{t}=t\end{matrix}$ 1a. Inverse $\begin{matrix}x=\bar{x}+vt\\ y=\bar{y}\\ z=\bar{z}\\ t=\bar{t}\end{matrix}$

We know that length and time will transform. The “how” they transform is shown: (replace bar with prime here)
Let's do a quick check: a particle travels along +x with a velocity 5 m/s with respect to a frame at rest (which contains a fixed meter stick). After 1 s, in the frame which is not moving, the particle is observed to be at $x=5\,m$. However in the moving frame which is the rest frame of the particle, the particle is observed to be at $\bar{x}=0$. So the question that these transformations answer is this: given coordinate and time measurements in one frame of reference, what are these measurements in another frame of reference.

We can also obtain the Galilean velocity transformations

$$\bar{v}_x=\frac{d\bar{x}}{d\bar{t}}=\frac{d(x-vt)}{dt}=\frac{dx}{dt}-v\frac{dt}{d\bar{t}}=v_x-v : v_x=\frac{dx}{dt}=\frac{d\bar{x}+vd\bar{t}}{dt}=\bar{v}_x+v$$

Where v_x would represent a velocity along the x axis measured by the observer in the fixed frame and this answers the question “what velocity does the moving observer then measure?”. For the inverse measurement, $\bar{v}_x$ represents a measurement made by the moving observer and this answers the question “what velocity does the fixed observer then measure?”

Again let's check this with a simple example: the fixed observer sees the moving frame to be moving along +x with a speed of 2 m/s. The fixed observer sees a particle moving along +x with a speed of 1 m/s. What velocity does the moving observer then measure? Intuitively the answer here is -1 m/s. This is easily seen to be the case since $\bar{v}_x=1-2=-1\,m/s$. In the moving frame, suppose the moving observer measured the particle to move at 1 m/s along +x. then the fixed observer measures a speed of 3 m/s, again verified by the inverse transformation.

Almost everyone uses prime notation to represent coordinates in special relativity. Replace the overlines now with primes.

2. Lorentz transformations:
$$\bar{x}=\gamma(x-vt) \quad \bar{y}=y \quad \bar{z}=z \quad \bar{t}=\gamma\left(t-\frac{v}{c^2}x\right)$$

2a. Inverse
$$x=\gamma(\bar{x}+v\bar{t}) \quad y=\bar{y} \quad z=\bar{z} \quad t=\gamma\left(\bar{t}+\frac{v}{c^2}\bar{x}\right)$$

In terms of primes, these become:
$$x'=\gamma(x-vt) \quad y'=y \quad z'=z \quad t'=\gamma\left(t-\frac{v}{c^2}x\right)$$
and
$$x=\gamma(x'+vt') \quad y=y' \quad z=z' \quad t=\gamma'\left(t+\frac{v}{c^2}x'\right)$$
.

So, let's check these out: suppose at t=0, t'=0. How does x transform? In other words, look at the Lorentz transformations and in the fixed frame you have x,y,z and t. What are their values in the moving frame? Decide if you need the transform or it's inverse. Take your lead here from the Galilean transformations.

Look at : $x'=\gamma(x-vt)$:Answer: $x=\frac{x'}{\gamma}$ ... in agreement with previous results.

suppose at x=0, x'=0. How does t transform? Look at $t=\gamma\left(t'+\frac{v}{c^2}x'\right)$

Answer: $t=\gamma t'$

What is this? This is the time in the rest frame (t) using t' measurements.

So; a meter stick at rest in Frank's frame is reported to be too short in Mary's frame (by Mary, looking at Frank's meter stick) and a clock in Mary's frame is observed to be slow (by Frank, looking at Mary's meter stick).

In the application of these Lorentz transformations, it is absolutely essential to get the details of the experiment laid out correctly. Who is measuring and what is being measured. For Frank to make a length measurement, he needs to have a meter stick in his frame and for Mary to measure time, she needs a watch in her frame.