	Physics 240: worksheet 21	Name: ________________

(1) Cowboy Ryan decides it's time to take a break from the dairy bar and takes a road trip to Cape Hatteras for a visit to the lighthouse (http://www.nps.gov/caha/index.htm). While there, he looks up and notices a very large spider dangling a mere 10 cm above his eye. He also notices that the massless silk thread connecting the spider is connected to the roof. Suddenly, he remembers my class and sets the spider into motion and times the period of oscillation.
(a) If the total length of the thread was 25 m, what period did he measure?
(b) Cowboy Ryan notices another spider up higher which is oscillating with a period of 5 s. How long is the second spider's thread?
(c) Now, suppose the first spider became scared of Cowboy Ryan's eye and extended itself another 2 m closer to the floor (Cowboy Ryan is very tall). What would the period of oscillation be now?

(2) On the moon, $g_{moon} = \frac{1}{6} g_{earth}$, a pendulum has a period of 1 s.
(a) What is the spring constant required for a mass of 1 kg to have the same period of oscillation?
(b) What is the length of the pendulum?
(c) What would be the period of this pendulum on the Earth?

(3) A mass m is placed inside a frictionless hemispherical bow with a radius of curvature R and given a small push up. Find the frequency (f) of small oscillations.

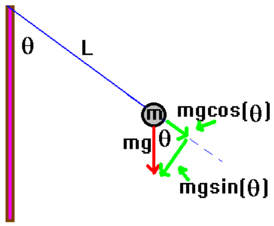
(4) Suppose a rod has a total length L=1 m and a mass m. The rod is hinged about one end. Find the period of oscillation for this physical pendulum. The moment of inertia about the end of the rod is given by $I = \frac{1}{3} mL^2$.

(5) Suppose a metal ring of radius R=1 m is hung from a peg. Find the frequency of small oscillations for this physical pendulum. The moment of inertia for the hoop about one end is given by $I = I_{cm} + mR^2 = 2mR^2$ (this involves application of a theorem known as the parallel axis theorem).

(6) A cylindrical cork bob with a cross sectional area A and height h is pulled a distance x down into the water from the equilibrium position. Find the restoring force on the cork and also the frequency of small oscillations about the equilibrium position.

A mathematical preliminary note: Small angle approximations

$$\sin(\theta) = \sum_{j=0}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \approx \theta \text{ when } \theta \text{ is small}$$



The simple pendulum

The simple pendulum consists of a point mass connected to a string of length L. The string does not stretch in this version. The system will undergo simple harmonic oscillation (SHO). Let's see how this comes about:

The torque about the point where the string is connected is given by:

$$\left|\vec{\Gamma}\right| = \left|\vec{R} \times \vec{F}\right| = mgL\sin(\theta).$$

As pictured, this torque is negative in direction since it produces a clockwise angular acceleration. Thus, $\Gamma = -mgL\sin(\theta)$. By Newton's laws, a torque results in an angular acceleration: $\Gamma = I\alpha$. The moment of inertia for a point mass which is at a distance L from a pivot point is given by $I = mL^2$. Let's put all this together:

$$\left[\Gamma = -mgL\sin(\theta)\right] = \left[I\alpha = mL^2\alpha\right] \Rightarrow \alpha = -\tfrac{g}{L}\sin(\theta).$$

You know, of course, that α and θ are related in pretty much the same way that a and x are related. On the face of it, however, the restoring force (here, a restoring torque) is not linear in the displacement variable although it is restoring. We fix this by the ***assumption of small angles***: we keep θ small enough so that $\sin(\theta) \approx \theta$. With this replacement, we find that we have

$$\alpha = -\tfrac{g}{L}\theta \text{ or } \tfrac{d^2\theta}{dt^2} + \tfrac{g}{L}\theta = 0.$$

The solutions to the angular equation are

$$\theta(t) = \theta_{max}\cos(\omega t + \varphi).$$

We can show this by taking the second derivative of our solution:

$$\tfrac{d}{d\theta}\left[\tfrac{d\theta(t)}{d\theta}\right] = \tfrac{d}{d\theta}\left[-\omega\theta_{max}\sin(\omega t + \varphi)\right] = -\omega^2\theta_{max}\cos(\omega t + \varphi)$$

This provides the condition required for the angular frequency:

$$\omega = \sqrt{\tfrac{g}{L}}$$

Note:

Now, you might like to see this written in terms of position variables. We can do this by using the relationship for arc length: $s = R\theta$ which here becomes $s = L\theta$. Then, the acceleration is seen through:

$$a = \alpha L = -\tfrac{g}{L}(L\theta) = -\tfrac{g}{L}s.$$

However you want to look at it, one thing stands out: the solution is described by:

$$\theta(t) = \theta_{max}\cos(\omega t + \phi) \text{ or } s(t) = A\cos(\omega t + \phi)$$

We need to be careful about the distinction between angular velocity and angular frequency. You'll see I've used Ω to represent the ***instantaneous angular velocity*** below.

$$\Omega = -\omega\theta_{max}\sin(\omega t + \phi) \text{ or } v(t) = -\omega A\sin(\omega t + \phi)$$

And

$$\alpha_{\text{(instantaneous angular acceleration)}} = -\omega^2\theta_{max}\cos(\omega t+\phi) \text{ or } a(t)=-\omega^2 A\cos(\omega t+\phi)$$

Presumably the difficulty that I have in choosing the symbols here leads to the reason that you won't often see these written in text books as I have done here. Again, what is the difficulty?

In any event, you can clearly see that for the simple pendulum, we have:

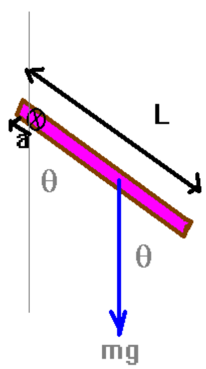
$$\omega=\sqrt{\tfrac{g}{L}},\ f=\tfrac{1}{2\pi}\sqrt{\tfrac{g}{L}},\ T=2\pi\sqrt{\tfrac{L}{g}}$$

The physical pendulum

The physical pendulum has a moment of inertia about a pivot which is at a distance “a” from the end as shown is uniform (with only a little bit more work, we could consider non-uniform systems) and has a total length L. We can use our formulation of the simple pendulum to find the frequency for small oscillations about the equilibrium.

The force of gravity is acting through the center of mass (i.e. at L/2) and is producing a torque about the indicated pivot which is given by:

$$\Gamma=-mg\left(\tfrac{L}{2}-a\right)\sin(\theta)$$



We'll keep a less than L/2 here. As before, we then have:

$$\Gamma=I\alpha$$

although here, we don't necessarily know the exact value for I. The good thing is that we might just be able to measure it!

Let's see how .... We equate the two expressions for torque:

$$-mg\left(\tfrac{L}{2}-a\right)\sin(\theta)=I\alpha \Rightarrow \alpha+\tfrac{mg}{I}\left(\tfrac{L}{2}-a\right)\sin(\theta)=0$$

Let's only worry about small oscillations here so that $\sin(\theta)\approx\theta$. This then gives:

$$\alpha+\tfrac{mg}{I}\left(\tfrac{L}{2}-a\right)\theta=0$$

If you want to write this in terms of a displacement variable (s), you would multiply by L.

In any event, you can recognize the form of this equation and you thus know that the equation describes simple harmonic oscillation with an angular frequency given by:

$$\omega=\sqrt{\tfrac{mg}{I}\left(\tfrac{L}{2}-a\right)}$$

Now, I promised you might be able to measure I ... here is how ....Suppose you know m, L, and a. Arrange your system so that you make measurements of the period, T. Then, $\omega=\frac{2\pi}{T}$. Next, invert this equation to find:

$$I=\frac{\omega^2}{mg\left(\frac{L}{2}-a\right)}$$

Thus, you are indeed able to measure the moment of inertia for the physical pendulum by a measurement of the period of oscillation. Note, however, that this is not I about the center of mass ... rather it is I about the pivot point. Also, it is interesting to note that if $I=mh(r)$ where r depends only upon the body geometry, then

$$\omega = \sqrt{\frac{g}{h(r)}\left(\frac{L}{2} - a\right)}$$

which is independent of the mass of the system, as one would expect for a simple pendulum.

(1) Cowboy Ryan decides it's time to take a break from the dairy bar and takes a road trip to Cape Hatteras for a visit to the lighthouse (http://www.nps.gov/caha/index.htm). While there, he looks up and notices a very large spider dangling a mere 10 cm above his eye. He also notices that the massless silk thread connecting the spider is connected to the roof. Suddenly, he remembers my class and sets the spider into motion and times the period of oscillation.
(a) If the total length of the thread was 25 m, what period did he measure?
(b) Cowboy Ryan notices another spider up higher which is oscillating with a period of 5 s. How long is the second spider's thread?
(c) Now, suppose the first spider became scared of Cowboy Ryan's eye and extended itself another 2 m closer to the floor (Cowboy Ryan is very tall). What would the period of oscillation be now?

Solution:
(a) If the total length of the thread was 25 m, what period did he measure?

$$\omega = \sqrt{\frac{g}{L}} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{25}{9.8}} = 10s$$

(b) Cowboy Ryan notices another spider up higher which is oscillating with a period of 5 s. How long is the second spider's thread?

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \left(\frac{T}{2\pi}\right)^2 = \frac{L}{g} \Rightarrow L = g\left(\frac{T}{2\pi}\right)^2 = 9.8\left(\frac{5}{2\pi}\right)^2 = 6.2m$$

(c) Now, suppose the first spider became scared of Cowboy Ryan's eye and extended itself another 2 m closer to the floor (Cowboy Ryan is very tall). What would the period of oscillation be now?

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{27}{9.8}} = 10.4s$$

(2) On the moon, $g_{moon} = \frac{1}{6}g_{earth}$, a pendulum has a period of 1 s.
(a) What is the spring constant required for a mass of 1 kg to have the same period of oscillation?
(b) What is the length of the pendulum?
(c) What would be the period of this pendulum on the Earth?

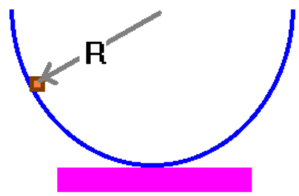
Solution:
We're really asked in the first part to find the period of a spring-mass system. Thus,

$$\text{(a)}\quad \omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} \Rightarrow k = m\left(\frac{2\pi}{T}\right)^2 = 1\left(\frac{2\pi}{1}\right)^2 = 39.5\frac{N}{m}$$

$$\text{(b)}\quad \omega = \sqrt{\frac{g_m}{L}} = \frac{2\pi}{T} \Rightarrow \left(\frac{2\pi}{T}\right)^2 = \frac{g_m}{L} \Rightarrow L = g_m\left(\frac{T}{2\pi}\right)^2 = \frac{9.8}{6}\left(\frac{1}{2\pi}\right)^2 = 0.041m$$

$$\text{(c)}\quad \omega = \sqrt{\frac{g}{L}} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{0.041}{9.8}} = 0.41s$$

(3) A mass m is placed inside a frictionless hemispherical bow with a radius of curvature R and given a small push up. Find the frequency (f) of small oscillations.



Solution:
It's easiest to draw the picture here. You can then see that the mass is constrained to be at a constant length R from the center. This is, then, the same condition as we have for the simple pendulum. Thus, the frequency of oscillation here is the same as for the simple pendulum. Thus,

$$\omega = \sqrt{\frac{g}{R}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{g}{R}}$$

(4) Suppose a rod has a total length L=1 m and a mass m. The rod is hinged about one end. Find the period of oscillation for this physical pendulum. The moment of inertia about the end of the rod is given by $I = \frac{1}{3}mL^2$.

Solution:
For the uniform physical pendulum, the moment of inertia is given by:

$$\omega = \sqrt{\frac{mg}{I}\left(\frac{L}{2} - a\right)}$$

For this system, $f(r) = \frac{1}{3}L^2$. Thus,

$$\omega = \sqrt{\frac{g}{\frac{1}{3}L^2}\left(\frac{L}{2} - a\right)} = \sqrt{3g\left(\frac{1}{2L} - \frac{a}{L^2}\right)} = \sqrt{\frac{3}{2}g} = 3.8\,\frac{\text{"rad"}}{s}$$

Thus: $f = \frac{1}{2\pi}\omega = \frac{3.8}{2\pi} = 0.61Hz$ and $T = \frac{1}{f} = \frac{1}{0.61} = 1.64s$

Notice that the human arm is approximately a rod 1 m long and pivoted about one end.

Here's an interesting problem (only for your interest … this problem won't be on a test).

Who's the more effective walker, a person with legs of length 1 m or a cat with legs of 0.1 m? If we stay within the limit of small oscillations, then we'll limit the swing for each to about 10^0. From problem 1, let's treat each of the legs as physical pendulums. The amplitude is restricted to 10 degrees. In one step, the entity moves forward by

$$s = L\theta = \begin{matrix} 1x10x\frac{2\pi}{360} = 0.17m & \text{person} \\ 0.1x10x\frac{2\pi}{360} = 0.017m & \text{cat} \end{matrix}$$

The angular frequency of each is given, as in problem (1) by:

$$\omega = \sqrt{\frac{g}{\frac{1}{3}L^2}\left(\frac{L}{2} - a\right)} = \sqrt{\frac{3g}{2L}}$$

We can find the period by

$$\omega = 2\pi f = \frac{2\pi}{T} = \sqrt{\frac{3g}{2L}} \Rightarrow T = 2\pi\sqrt{\frac{2L}{3g}}$$

The periods are given by:

$$T = 2\pi\sqrt{\frac{2L}{3g}} = \begin{matrix} 2\pi\sqrt{\frac{2}{3g}} = 1.64s & \text{person} \\ 2\pi\sqrt{\frac{0.2}{3g}} = 0.52s & \text{cat} \end{matrix}$$

Well let's find the velocity for each:

$$v_{person} = \frac{s}{T} = \frac{0.17m}{1.64s} = 0.10m/s$$
$$v_{cat} = \frac{s}{T} = \frac{0.017m}{0.52s} = 0.033m/s$$

It seems then with this simple approximation that indeed people are about 3 times more effective in crossing a given distance than cats are … however each is restricted to walk at a "natural" gait which means that the legs swing only at the natural frequency of oscillation. In reality, other things can happen (dogs, mice, etc). Speaking of mice … what's the natural velocity for movement for mice if their legs are only about 0.01m long? The same analysis will lead to T=0.16s for mice and V=0.011m/s! (cats move about 3x faster than mice under these approximations). Probably, restricting the angle to 10 degrees is too small. Also, there is much more to this type of dynamics than this problem describes.

(5) Suppose a metal ring of radius R=1 m is hung from a peg. Find the frequency of small oscillations for this physical pendulum. The moment of inertia for the hoop about one end is given by $I = I_{cm} + mR^2 = 2mR^2$ (this involves application of a theorem known as the parallel axis theorem).

Solution:
This system is not uniform but we can still deal with it since we know where the center of mass of the hoop is. Thus, the frequency for small oscillations is given by:

$$\omega = \sqrt{\frac{mg}{I}\left(\frac{L}{2} - a\right)} \Rightarrow \omega = \sqrt{\frac{g}{2R^2}R} = \sqrt{\frac{g}{2R}}$$

(note: the hoop has a total "length" of 2R which is the diameter, in the sense that we've used the variable L above.
Thus:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{2R}} = 0.352Hz, T = \frac{1}{f} = 2.84s$$

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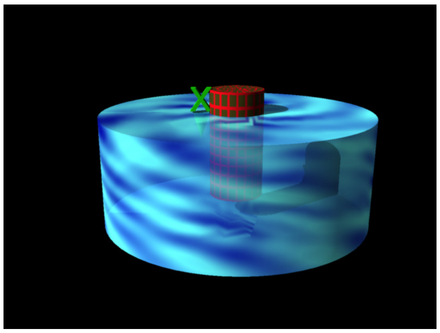
(6) A cylindrical cork bob with a cross sectional area A and a height h is pulled a distance x down into the water from the equilibrium position. Find the restoring force on the cork and also the frequency of small oscillations about the equilibrium position.

Solution:



The extra buoyant force on the cork is given by the weight of the fluid displaced. Thus, we have:

$$F_b = -\rho_{water} g(Ax)$$

here, the – sign indicates upward. This produces an acceleration given by:

$$m_{cork} a = \rho_{cork}(Ah)a$$

Let's put the two together to find:

$$a(Ah)\rho_{cork} = -\rho_{water} g(Ax)$$

More simply, this appears as:

$$a = -\frac{\rho_{water}}{\rho_{cork}}\frac{g}{h}x$$

The cork will undergo simple harmonic oscillation with a frequency given by:

$$\omega = \sqrt{\frac{\rho_{water}}{\rho_{cork}}\frac{g}{h}}$$

This is only a first order approximation since damping forces are quite large here.