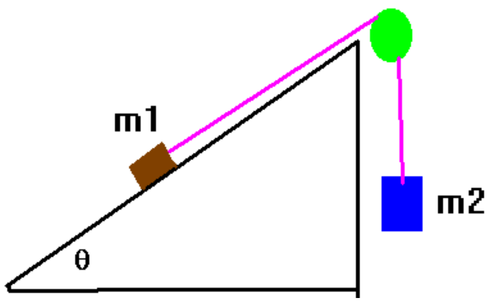
	Physics 240: Worksheet 09	Name ________________

Don't forget: discussion of Newton's third law and bugs (F_{12}=-F_{21}).

(1) A mass is on an inclined plane as shown. Find the acceleration and tension of the system in case it is accelerating towards the right. The coefficient of friction here is μ.



Then, if a=0, find the required condition for μ.

(2) Work defined. Suppose a mass at rest is lying on a frictionless table and a force F is applied to the mass parallel to the surface of the table through a distance Δx. What is the change in velocity of the mass?

(3) A force F is applied to a mass M which is just equal to only a slight amount more than the mass (but the difference is not measurable). The mass is lifted through a height Δy with a constant velocity.
(a) What is the work done by F?
(b) Did this result in a change in K?
(c) Where did this work go to?
(d) Is the work energy theorem general enough?
(e) Define total mechanical energy (E) and show that if E is conserved, we have a new tool for solving problems.

(4) Distinguish between non-conservative systems and conservative systems. Then, provide a more general definition for work.

(5) Hooke's law states F=-kx (the more you compress a spring, the more the spring presses back and in the opposite direction). Find the work required to compress the spring through a distance Δx. Note: k is the spring constant measured in N/m in the SI system. You will measure this in lab.

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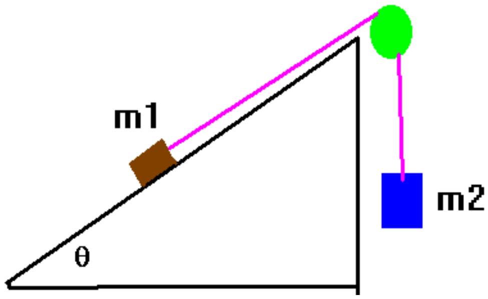
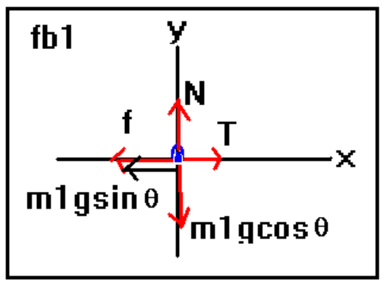
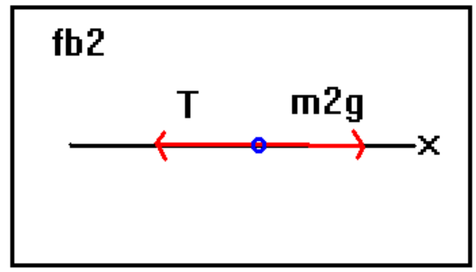
(1) A mass is on an inclined plane as shown. Find the acceleration and tension of the system in case it is accelerating towards the right. The coefficient of friction here is μ.



Then, if a=0, find the required condition for μ.

Solution: the two free body diagrams are shown below:





Where I have "unbent" the diagram for mass m2. We apply Newton's laws to this, assuming an acceleration in the +x direction:

$$m_2 g - T = m_2 a$$
$$\sum \vec{F} = m\vec{a} \Rightarrow T - \mu m_1 g\cos(\theta) - m_1 g\sin(\theta) = m_1 a$$
$$N - m_1 g\cos(\theta) = 0$$

We eliminate the tension by addition to give the acceleration:

$$a = g\left(\frac{m_2 - m_1(\mu\cos(\theta)+\sin(\theta))}{m_1+m_2}\right)$$

We can then also easily find the tension:

$$T = m_2 g - m_2 g\left(\frac{m_2 - m_1(\mu\cos(\theta)+\sin(\theta))}{m_1+m_2}\right)$$

Now, let's look at the special case a=0. In this case, we have the following results:

$$T - \mu m_1 g\cos(\theta) - m_1 g\sin(\theta) = 0$$
$$m_2 g - T = 0$$

Tension can be eliminated by addition:

$$m_2 g - \mu m_1 g\cos(\theta) - m_1 g\sin(\theta) = 0$$

Simplifying, we get:

$$m_2 - \mu m_1\cos(\theta) - m_1\sin(\theta) = 0 \text{ or}$$
$$\frac{m_2}{m_1} - \mu\cos(\theta) - \sin(\theta) = 0$$

Solving for μ, we get: $\frac{m_2}{m_1} - \sin(\theta) = \mu\cos(\theta) \Rightarrow \mu = \frac{\frac{m_2}{m_1} - \sin(\theta)}{\cos(\theta)}$.

(2) Work defined. Suppose a mass at rest is lying on a frictionless table and a force F is applied to the mass parallel to the surface of the table through a distance Δx. What is the change in velocity of the mass?

According to Newton's laws, $\vec{F} = m\vec{a} = m\frac{\Delta \vec{v}}{\Delta t}$. Let's look at the 3rd equation of motion, namely $v^2 = v_i^2 + 2a(\Delta x)$. Let's solve this for acceleration: $a = \frac{v^2 - v_i^2}{2\Delta x}$. Now, use this in Newton's law to give (we drop vectors here): $F = m\left(\frac{v^2 - v_i^2}{2\Delta x}\right)$. Multiply the top and the bottom by Δx to give: $F(\Delta x) = \frac{1}{2}m(v^2 - v_i^2)$. The term F(Δx) is given a special name: it is the **external work (W)** performed. Likewise, the term $\frac{1}{2}mv^2$ is also given a special name: it is the **kinetic energy (K)** of the system. We thus have the development of a fundamental theorem which is the work – energy theorem:

External work = change in kinetic energy.

Symbolically, we have: $W = \Delta K$.

In the present problem, we need to calculate each side of this equation and put the two results together: as I show below:

(1) Calculate Work: W=FΔx.

(2) Calculate ΔK: $\Delta K = K_f - K_i = \frac{1}{2}mv^2$.

(3) Put the two together: $F\Delta x = \frac{1}{2}mv^2$

(4) Solve for the velocity: $v = \pm\sqrt{\frac{2F\Delta x}{m}}$

The correct solution here is the + solution since the force is in the +x direction.

One subtle thing here: I've used the concept that there is such a thing as an initial and a final kinetic energy. Within a constant frame of reference, in potential free regions of space, this would be true in the absence of non-conservative forces. However, what is measured as K in one frame of reference is not the same in another frame. However, at low speeds, the change in kinetic energy is going to be invariant in an inertial reference frame. This is my disclaimer which does take this many words to say.

There are lots of sign conventions one can also use: work can be done by a system or work can be done on a system. Also, work can be done by a force in the direction of displacement or work can be done by a force in the opposite direction of displacement. You can thus see that there are 4 possible outcomes. As I am defining work above, and also below, I am defining the work done on a system by an external entity which is exerting a force $\vec{F}$ on the system while the system is undergoing a displacement $\Delta\vec{s}$ as positive and it is given by: $W = \vec{F} \bullet \Delta\vec{s}$. As an example, work done by the gravitation force in moving an object through a distance h: ($\Delta y = y_f - y_i = -h$) is a positive quantity:

$$W = \left[-mg\,\hat{j}\right] \bullet \left[-h\,\hat{j}\right] = +mgh$$

However, suppose that the object was thrown upward from the ground through a distance h to the apex of its trajectory. The work done by the gravitational force during this interval is given by:

$$W = \left[-mg\,\hat{j}\right] \bullet \left[+h\,\hat{j}\right] = -mgh$$

If, however, you asked what the work the object was doing on the entity producing the gravitational force, the signs are exactly opposite. For our purposes, I'll be concentrating on the work done upon an object by an external entity, just in case I get too short in my descriptions.

(3) A force F is applied to a mass M which is just equal to only a slight amount more than the mass (but the difference is not measurable). The mass is lifted through a height $\Delta y=h$ with a constant velocity.
(a) What is the work done by F?
(b) Did this result in a change in K?
(c) Where did this work go to?
(d) Is the work energy theorem general enough?
(e) Define total mechanical energy (E) and show that if E is conserved, we have a new tool for solving problems.
Solution:
(a) $W=F\Delta y=mgh$.
(b) no
(c) This work went into a term mgh.
(d) no
(e) Total mechanical energy is E=U+K. In physics, this is a conserved quantity. The meaning of this is the following equation:

$$\Delta E = 0 \Rightarrow U_i + K_i = U_f + K_f \,.$$

U is the symbol that we'll use for potential energy. For problems involving gravity close to the surface of the Earth (or, to be more precise, over relative short distances), we have that U=mgh. Let's show how this can be used to solve free-fall problems:
(i) A mass m at rest falls through a distance h. Find the velocity at this point.
(1) Calculate the changes:

$$U_i = mgh \; U_f = 0$$
$$K_i = 0 \; K_f = \tfrac{1}{2}mv^2$$

(2) then apply the conservation of energy:
$U_i + K_i = U_f + K_f \Rightarrow mgh + 0 = 0 + \frac{1}{2}mv^2 \Rightarrow v = \pm\sqrt{2gh}$. The correct solution here is -.
(ii) A mass is thrown straight up with an initial velocity v. How high does the mass go?
(1) Calculate the changes:

$$U_i = 0 \; U_f = mgh$$
$$K_i = \tfrac{1}{2}mv^2 \; K_f = 0$$

(2) then apply the conservation of energy:

$$U_i + K_i = U_f + K_f \Rightarrow 0 + mgh = 0 + \tfrac{1}{2}mv^2 \Rightarrow h = \tfrac{v^2}{2g}$$

Now actually the expression that we've used for the gravitational potential energy is really only valid for rather short distances. The actual more complicated expression is given by Newton's law of gravitation which says that for two bodies of mass m_1 and m_2 which are separated by a distance r, the gravitation potential energy would be given by:

$$U = -G\frac{m_1 m_2}{r}\,.$$

Here G is called the "universal gravitational constant" and has the value $6.67\text{x}10^{-11}$ Nm^2/kg^2. You can also, from the gravitation force which looks like: $\left|\vec{F}\right| = G\frac{m_1 m_2}{r^2}$ obtain the acceleration due to gravity: $G\frac{m_1 m_2}{r^2} = m_2 g \Rightarrow g = G\frac{m_E}{r_E^2}$ by realizing that this is the gravitation force near the surface of the Earth.

(4) Distinguish between non-conservative systems and conservative systems. Then, provide a more general definition for work.

In conservative systems, energy is conserved *to within all the variables we are considering.* This also means that the work done depends only upon the initial and final states of the system. In non-conservative systems, ***we need to consider the particular path taken by the system.*** Although energy is still conserved, our equations fail to consider some of the forms of energy. Perhaps the best example of a non-conservative system is the work done by a frictional force. Our equations of motion may not consider the formation of heat which we will show later is a form of kinetic energy. How much heat is evolved depends upon the path. We modify the work-energy theorem for the presence of non-conservative forces to appear as:

$$\Delta K_{non-conservative} = \Delta U + \Delta K_{conservative}.$$

Later I'll show you how to use this.

Work is more generally defined for a variable force which has a direction in space as: $W = \sum \vec{F}_i \bullet \Delta\vec{s}_i$ where the sum means that you can add up small amounts to approximate the total work. Here, s represents the path (or displacement) and the "dot" is a dot product. Notice that the work can be positive (F and s in the same direction), negative (F and s in opposite directions) or zero (F and s are at right angles or the initial point and the final point are the same for a conservative system).

(5) Hooke's law states F=kΔx (the more you compress a spring, the more the spring presses back and in the opposite direction). Find the work required to compress the spring through a distance Δx if the spring was initially uncompressed (more on this detail will come on the next worksheet).
Solution:

There is one more important detail about Hooke's law: the coordinates in Hooke's law really must refer strictly to an amount of compression or expansion. If Hooke's law simply relied upon the coordinate, ridiculous expressions would result. For example, the spring would exert more force at one position in space than another point in space. This means that the Δx that you see in this means a "change in the spring length" and does not necessarily refer to the coordinate of the end of the spring.

We can determine the work done for a "linear" force such as this by determination of the average force which the spring exerts: $F_{average} = \frac{1}{2}k(\Delta x)$ provided the initial compression and the initial force are both zero. The work to compress the spring is then given by W=F$_{average(}\Delta$x). If the initial x position is zero, then we have $W = \frac{1}{2}k(\Delta x)^2$. If you now want to refer this strictly to a spatial coordinate, the work would be given by:

$$W = \frac{1}{2}k(\Delta x)^2$$

But, we have calculus here so let's do this in the calculus way:

$$W = \oint \vec{F} \bullet d\vec{s} = \oint F d(\Delta x) = \int_{\Delta x_i}^{\Delta x_f} k(\Delta x) d(\Delta x) = \int_{0}^{\Delta x} k(\Delta x) d(\Delta x) = \frac{1}{2}k(\Delta x)^2$$

Now, the next question is what is the potential energy of a spring which was initially at zero compression and was then compressed by an amount Δx. The answer is $U_{spring} = \frac{1}{2}k(\Delta x)^2$. Suppose a mass m is placed on the compressed spring. After the spring is released, how fast will the mass move? Answer: Apply conservation of energy:

$$U_i = \frac{1}{2}k(\Delta x)^2 \quad U_f = 0$$
$$K_i = 0 \quad K_f = \frac{1}{2}mv^2 .$$

This gives: $U_i + K_i = U_f + K_f \Rightarrow \frac{1}{2}kx^2 + 0 = 0 + \frac{1}{2}mv^2 \Rightarrow v = \pm(\Delta x)\sqrt{\frac{k}{m}}$. Which of the signs is chosen depends upon the direction the spring was initially compressed. As a modification, suppose the spring were pointed upward. The spring is compressed through a distance Δx and then released. What is the speed that the mass has at the instant it leaves the spring?
The answer is not so simple mathematically but can still be solved:

$$\frac{1}{2}k(\Delta x)^2 = mg(\Delta x) + \frac{1}{2}mv^2 \Rightarrow v = \pm\sqrt{\frac{k}{m}(\Delta x)^2 - 2g(\Delta x)}$$

One final note: be careful not to confuse K and k in these problems.