	Physics 210: Worksheet 24	Name: ________________

(1) Cowboy Ryan is on the road again! Suppose that he is inside one of the many caverns that are found around the Whitehall area of Montana (which is also, by the way, close to Wheat Montana). He notices that in one of the caverns that he is standing in resembles a large open-ended organ pipe with a length of about 1000 m. What modes of oscillation would it be possible to hear? You may assume that the speed of sound is 343 m/s.

(2) Katie is visiting a distant silo. With her new-found knowledge of standing waves and the like she immediately investigates modes of oscillation in that silo. Suppose the silo was 100 m tall, and a recent unfortunate tornado had torn the top right off of it so that it was exposed to the atmosphere on the upper end. What modes of oscillation would you expect the silo to vibrate in? You may assume the speed of sound is 343 m/s.

(3) Suppose that the sororities and fraternities did not get along so well on the Lyon campus. After a most recent episode, one of the sororities decided to declare war on one of the frat houses. After leaving physics one day, together with an extensive internet search involving the effects of ultra-low frequency sound on the human body, it was decided to construct a debunked weapon capable of producing a sound wave of 0.1 Hz in the shortest possible space and directing it towards a frat house. What type of design should this have and how long should it be? You may assume the speed of sound is 343 m/s.

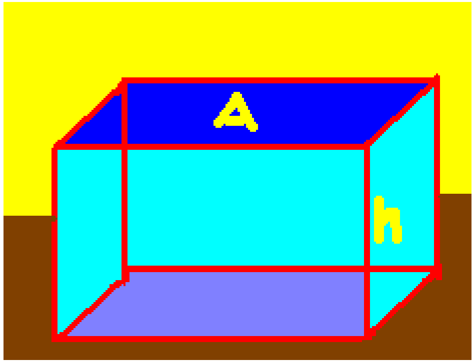
(4) Hope is on the white cliffs of Dover contemplating whether or not to take a trip through the Chunnel. If the Chunnel has a total length of 31 miles or about 50,000 m and the speed of sound is 360 m/s, what modes of oscillation might the Chunnel have?

(5) Speaking of the Chunnel, it turns out that the under water parts are at an average depth of 150 ft below sea level or about 46 m below the ocean. What is the gauge and absolute pressure at that point.

Sound and Standing Waves of Sound

In order to understand sound, you will need to understand pressure and molecular motion. Let's begin with pressure.

Imagine an aquarium which has a cross sectional area A and a height h. Also, for the sake of argument, imagine that the walls of the aquarium don't contribute a significant mass to this problem.



The first question is this: Suppose the aquarium is filled completely with water. What is the weight of the water on the table?

$$Weight = m_{water} g = \rho_{water} g V_{water} = \rho_{water} g (Ah)$$

Now, pressure is defined as force per unit area. If we as what the pressure at the bottom of the aquarium is, then we find:

$$P \equiv \frac{Force}{Area} = \frac{\rho g A h}{A} = \rho g h$$

So, a column of water of height h will produce a pressure proportional to the height of the column. The shorter the column, the less the pressure. In the SI system of units, we measure pressure in units of Pascals or Pa which are given by:

$$1\,Pa = \frac{1\,N}{1\,m^2}$$

We measure pressure in several ways ... we can measure ***gauge pressure*** or ***absolute pressure***. The difference between these methods of measurement essentially means that if you measure gauge pressure, you set your zero of your gauge to not measure, for example, the weight of the column of air from the atmosphere .. you would only be measuring differences from the ambient atmospheric pressure. If you included the atmospheric pressure in these measurements you would be measuring absolute pressure. For much of our work, we'll be using gauge pressure.

We will soon need to use these concepts in order to discuss thermodynamics in addition to sound.

When a sound pulse propagates through the atmosphere, in the linear approximation, air molecules move about a fixed equilibrium position but on the average, stay fixed. The molecules bump against each other and transmit energy. The types of waves which these are is described as longitudinal. We imagine a long cylinder of air which bounds the system with a speaker vibrating at one end. I've provided an animated gif showing the compressions and rarefactions in a column of air as I have described.

Now to understand sound waves, and in particular standing sound waves, it is important to realize that we can plot two different things: (1) molecular displacement or (2) pressure amplitudes. Which of these you plot makes the world of difference in your picture.

The molecular displacement is 90 degrees out of phase with the pressure amplitude.

We can prove this, although it's just a bit complicated. Here is how to do this:
Suppose the motion of the piston is given by:

$$s(x,t)=s_{max}\cos(\omega t)$$

This will push off the molecules and what will result is longitudinal traveling waves given by:

$$s(x,t)=s_{max}\cos(kx-\omega t)$$

How can this come about?
In order to describe this, we first define the Bulk Modulus (B) of a material as:

$$B\equiv\frac{-\Delta P}{\left(\frac{\Delta V}{V}\right)}$$

Here, P is pressure, and V is volume and the - sign means things get smaller as P increases.

The Bulk Modulus answers the following question:
Suppose I have a volume V of a material and I subject this material to a change in pressure ΔP. How much will the volume of the material change as a result?

We can rewrite the definition of B in another way:

$$\Delta P=-B\left(\frac{\Delta V}{V}\right)$$

For our original pulse, suppose that we consider an initial volume of the gas of cross sectional area A with length L=Δx. The initial volume of the gas will then be given as

$$V=A(\Delta x)\ .$$

By compression, this gas will get squeezed into perhaps a smaller volume (or rarified into a larger volume). The change in volume we'll represent in terms of molecular movement along the x-axis (Δs) as:

$$\Delta V=A(\Delta s)$$

The Bulk Modulus definition then becomes:

$$\Delta P = -B\left(\frac{\Delta s}{\Delta x}\right)$$

We can also find the "longitudinal vibration velocity" using calculus as

$$v_{long} = \omega s_{max} \sin(kx - \omega t)$$

Thus, the pressure variation is given by

$$\Delta P = B k s_{max} \sin(kx - \omega t) = (\Delta P_{max}) \sin(kx - \omega t)$$

and the "Longitudinal vibration velocity" is:

$$v_{long} = \omega s_{max} \sin(kx - \omega t)$$

The pressure pulse is 90^0 out of phase with the molecular displacement!

It would lead us a bit astray to derive the speed of sound so I'll quote the result. (this is best done with thermodynamics which we have not studied yet)

$$v = \sqrt{\frac{B}{\rho}} \Rightarrow B = v^2 \rho$$

You also know that for a traveling harmonic wave:

$$\frac{\omega}{k} = v$$

This allows us to solve for the pressure amplitudes:

$$\Delta P_{max} = B \frac{\omega}{v} S_{max} = \rho v \omega s_{max}$$

We are now also in a position to obtain the energy contained in a small region of the gas:

$$\Delta K = \frac{1}{2} m v_{longitudinal}^2 = \frac{1}{2} (\rho A \Delta x)(\omega^2 s_{max}^2)(\sin^2(kx - \omega t))$$

Now, you might remember the "IRS" argument that we did for the string. I know it's probably a little bit harder to think of air as having elasticity but if this argument is ever going to give the right value, you pretty much have to consider that the total time – average energy is partially due to potential energy. Thus, applying the "IRS" argument, we get:

$$< \Delta E > = \frac{1}{2} (\rho A)(\omega^2 s_{max}^2) \Delta x$$

Now, it's actually the power which we are interested in. Thus:

$$< Power > = < \frac{\Delta E}{\Delta t} > = \frac{1}{2} \rho A (\omega s_{max}^2) v$$

As it turns out, when we're talking about sound waves or electromagnetic waves, we're often more interested in the intensity (I) (in SI units … W/m^2), (valid for a detector normal to the incident energy):

$$I \equiv \frac{< Power >}{Area}$$

For this sound wave, the intensity is then given by:

$$I=\frac{1}{2}\rho\left(\omega s_{max}\right)^2 v=\frac{\Delta P_{max}}{2\rho v}$$

Standing waves in organ pipes

Back to my original statement … there are 2 ways to plot standing waves in an organ pipe, either as pressure plots or as molecular displacement plots. You've got to understand the difference between the two.

We're going to consider 2 boundary conditions.

(1) At a rigid surface:

$$s=0$$
$$\Delta P=\Delta P_{max}$$

The easiest way to see this is by realizing first that molecules don't travel through rigid surfaces, and then there is the 90^0 phase difference.

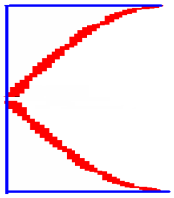
(2) At a free surface (this is only an approximate boundary condition):

$$s=s_{max}$$
$$\Delta P=0$$

The easiest way to imagine this is that since P=F/A, inside the tube, A is small while outside the tube, A is large. The easiest way to avoid a discontinuity in P is by saying P=0. In fact, the area exposed to sound is more like a cone extending from the end of the tube (not infinite), which makes this only approximate (think of the wake of a duck in a pond).

	Physics 210: Worksheet 24	Name: _______________

In the plots of sound waves that I will plot, we'll use plots of molecular displacement vs. position.



(1) Closed organ pipes (= 1 end closed, 1 end open)

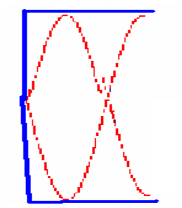
Consider an organ pipe which has 1 end open and 1 end closed.
We have mixed bc, and thus the shortest wavelength of a standing wave which can be established is:

$$\frac{1}{4}\lambda_1 = L \Rightarrow \lambda_1 = 4L$$

This gives us the lowest frequency:

$$f_1\lambda_1 = v \Rightarrow f_1 = \frac{v}{4L}$$

The next highest mode of oscillation has:



$$\frac{3}{4}\lambda_3 = L \Rightarrow \lambda_3 = \frac{4}{3}L$$

where I've again used 3 instead of 2 for reasons which ought to be obvious by now.
The corresponding frequency is then given by:

$$f_3\lambda_3 = v \Rightarrow f_3 = 3\left(\frac{V}{4L}\right)$$

It's easy to see for this organ pipe, the general solution is:

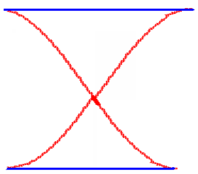
$$f_n^{closed} = n\left(\frac{v}{4L}\right); n = 1,3,5,...$$

We say that the even modes of oscillation are not present here.
Notice that you can write this last expression as:

$$f_n^{closed} = (2n-1)\left(\frac{v}{4L}\right); n = 1,2,3,...$$

and you would say that the even harmonics are absent for this system.

(2) Open organ pipes (=both ends of a tube are open)

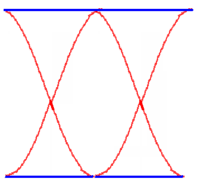


The lowest mode of oscillation has one node and an antinode at each end. Thus, the wavelength is given by:

$$\frac{1}{2}\lambda_1 = L \Rightarrow \lambda_1 = 2L$$

The lowest lying frequency is then given by:

$$f_1\lambda_1 = v \Rightarrow f_1 = \frac{v}{2L}$$



The next to lowest mode of oscillation has an antinode at each end and 2 nodes. Thus, the wavelength is given by:

$$\lambda_2 = L$$

The next to lowest mode of oscillation is then given by:

$$f_2\lambda_2 = v \Rightarrow f_2 = \frac{v}{L} = 2\left(\frac{v}{2L}\right)$$

At this point, it's no big stretch of the imagination to realize that for these organ pipes, the general resonance condition is given by:

$$f_n^{open} = n\left(\frac{v}{2L}\right); n = 1,2,3,\ldots$$

(1) Cowboy Ryan is on the road again! Suppose that he is inside one of the many caverns that are found around the Whitehall area of Montana (which is also, by the way, close to Wheat Montana). He notices that in one of the caverns that he is standing in resembles a large open-ended organ pipe with a length of about 1000 m. What modes of oscillation would it be possible to hear? You may assume that the speed of sound is 343 m/s.

Solution:
For the open-ended organ pipe, the resonance condition is given by:

$$f_n^{open} = n\left(\frac{v}{2L}\right); n = 1,2,3,...$$

Here, L=1000 m, v=343 m/s. Thus,

$$f_n = n\left(\frac{343}{2000}\right) = 0.1715 \text{ Hz}$$

The first 3 modes are then given by:

$$f_1 = 0.1715 \text{ Hz}$$
$$f_2 = 0.343 \text{ Hz}$$
$$f_3 = 0.5145 \text{ Hz}$$

(2) Katie is visiting a distant silo. With her new-found knowledge of standing waves and the like she immediately investigates modes of oscillation in that silo. Suppose the silo was 100 m tall, and a recent unfortunate tornado had torn the top right off of it so that it was exposed to the atmosphere on the upper end. What modes of oscillation would you expect the silo to vibrate in? You may assume the speed of sound is 343 m/s.

Solution:
After the tornado came through, the silo was approximately a closed organ pipe by our definition of closed (meaning 1 end closed, 1 end open). Thus, the frequencies of oscillation are given by:

$$f_n^{closed} = n\left(\frac{v}{4L}\right); n = 1,3,5,...$$

Here, L=100 m, so the frequencies are:

$$f_n = n\left(\frac{343}{400}\right) = n(0.8575)$$

The lowest 3 modes of oscillation are:

$$f_1 = 0.8575 \text{ Hz}$$
$$f_3 = 2.5725 \text{ Hz}$$
$$f_5 = 4.2875 \text{ Hz}$$

(3) Suppose that the sororities and fraternities did not get along so well on the Lyon campus. After a most recent episode, one of the sororities decided to declare war on one of the frat houses. After leaving physics one day, together with an extensive internet search involving the effects of ultra-low frequency sound on the human body, it was decided to construct a debunked weapon capable of producing a sound wave of 0.1 Hz in the shortest possible space and directing it towards a frat house. What type of design should this have and how long should it be? You may assume the speed of sound is 343 m/s.

Solution: we can compare the lowest mode of oscillation from open and closed organ pipes:

$$\frac{f_1^{open}}{f_1^{closed}}=\frac{1\left(\frac{v}{2L^{open}}\right)}{1\left(\frac{v}{4L^{closed}}\right)}=\frac{4L^{closed}}{2L^{open}}=2\frac{L^{closed}}{L^{open}}=1\Rightarrow L^{open}=2L^{closed}$$

This means, given two organ pipes which produce the same frequency, the shorter one is the closed (1 end closed, 1 end open) organ pipe. Now, how long is the pipe?

$$f_1^{closed}=\frac{v}{4L^{closed}}\Rightarrow L^{closed}=\frac{v}{4f_1}=\frac{343}{4(0.1)}=857\text{ m}$$

(4) Hope is on the white cliffs of Dover contemplating whether or not to take a trip through the Chunnel . If the Chunnel has a total length of 31 miles or about 50,000 m and the speed of sound is 360 m/s, what modes of oscillation might the Chunnel have?

Solution:
The Chunnel has 2 open ends so we'll use the results for the open-ended organ pipe here (this is only approximate since the Chunnel is curved):

$$f_n^{open}=n\left(\frac{v}{2L}\right);n=1,2,3,...$$

In this case, we then have:

$$f_n=n\left(\frac{360}{2\times\left(5\times10^4\right)}\right)=n\left(3.6\times10^{-3}\right)$$

The absolute lowest mode of oscillation will then be 0.0036 Hz. Probably these oscillations will not exist due to absorption of energy in the Chunnel walls.

(5) Speaking of the Chunnel, it turns out that the under water parts are at an average depth of 150 ft below sea level or about 46 m below the ocean. What is the gauge and absolute pressure at that point.

Solution:

$P = \rho g h = (1000)(9.8)(46) = 4.5 \times 10^5 \text{ Pa}$

Let's assume the density of sea water is $r = 1000 \frac{kg}{m^3}$. The gauge pressure is given by:

Now to get the absolute pressure, you need to add onto this the atmospheric pressure which is given by $P_{atmospheric} = 1 \times 10^5 \text{Pa}$. So the absolute pressure is 5.5×10^5 Pa.

I hope to have enough time to demonstrate a speed of sound measurement to you today.