

the twin paradox and Muons

Consider the twins Frank and Mary.

Mary is, at birth, placed on a spaceship and sent off to a planet which is 10 ly distant at a speed of $v = \beta c$. Upon arrival at the planet, the spaceship instantly turns around and returns to earth. How old are each of the twins upon the reunion?

It is easy to calculate Frank's age because this is given by:

$$\text{distance} = \text{rate} \times \text{time} \Rightarrow \Delta t_f = 2 \times \frac{10 \text{ly}}{\beta c} = \frac{20}{\beta} \text{yr}$$

$$\text{For example, if } \beta = 0.5 \Rightarrow \Delta t_f = 40 \text{yr}$$

Now, let's see about Mary's age. According to the Lorentz transformation,

$$\Delta t_m = \frac{1}{\gamma} (\Delta t_f)$$

We need to look at the meaning here, however. What this says is that an increment of time in a fixed frame becomes larger in a moving frame. This is quickly remembered (and I recommend this as a check of your results) by "moving clocks run slow."

Now if that is the case, we have in particular that 1 year according to Frank will only result in a time of $\frac{1}{\gamma}$ years for Mary. With this in mind, we're ready to calculate the time differences.

Now, Mary is the one that gets accelerated, not Frank. This means that the situation is not exactly symmetric: although for most of the trip, Mary and Frank see each other moving away or towards in a symmetric manner, the accelerations ultimately break this symmetry so that Frank is in the proper time frame.

When the two are again reunited, what is their age difference?

This is easy to obtain:

$$\text{age difference} = \Delta t_f - \Delta t_m = \frac{20}{\beta} - \frac{1}{\gamma} \left[\frac{20}{\beta} \right] = 20 \left(\frac{1}{\beta} \right) \left[1 - \sqrt{1 - \beta^2} \right]$$

We can easily calculate this for specific velocities. Here is an example. Suppose

$$\beta = 0.99 \Rightarrow \text{agedifference} = 20.20 [1 - 0.141] = 17.2 \text{yr}$$

Ok, so Frank would be 20.2 years old while Mary would only be 3 years old.

Here is another example that is a bit closer to reality. Muons are generated in the upper atmosphere by cosmic rays. In the rest frame of the Muons, the half-life of muons is $\tau = 1.52 \times 10^{-6}$ s. Suppose a cosmic ray burst created N_0 muons with $\beta = 0.98$. What we want to do is to determine how this number drops off as you go closer to the surface of the earth.

To do this, we need to understand how the number drops off in the rest frame of the Muons. The decay of muons is governed by exponential decay of the form:

$$N = N_0 e^{-\frac{t}{h}}$$

Let's find out what h is by looking at the time for half of the muons to decay. This is given by:

$$\left[\frac{N}{N_0} \right] = \frac{1}{2}$$

So:

$$\frac{1}{2} = e^{-\frac{t_{1/2}}{h}} \Rightarrow \ln\left(\frac{1}{2}\right) = -\frac{t_{1/2}}{h} \Rightarrow \ln(1) - \ln(2) = -\frac{t_{1/2}}{h} \Rightarrow h = \frac{t_{1/2}}{\ln(2)}$$

So we can write this as:

$$N = N_0 e^{-\frac{\ln(2)t}{\tau}}$$

where τ is the time it takes for the number to decay by $\frac{1}{2}$.

In the rest frame of the muons, the muons can travel for a distance of

$$L_0 = \beta c \tau$$

by the time $\frac{1}{2}$ of them have decayed. But in the rest frame of the Earth, the time changes as

$$t = \gamma \tau \Rightarrow L = \beta c (\gamma \tau)$$

(again, moving clocks run slow so the half-life in the Earth's frame is much larger).

In particular, let's calculate these distance for the specific numbers we have.

$$L_0 = 0.98c(1.52 \times 10^{-6}) = (1.49 \times 10^{-6}) \times (3 \times 10^8) = 447\text{m}$$

$$L = \gamma L_0 = \frac{1}{\sqrt{1-\beta^2}} L_0 = 5.03 L_0 = 2246\text{m}$$

The important point here is this: a greater number of muons would be detected closer to the earth than would be predicted by the classical calculations. According to the muons, the length is contracted. According to the earth-bound observer, the time is dilated to provide ultimate agreement between the two frames.

Now if these experiments were done, probably the easiest way would be to simply place detectors at several different altitudes and determine the experimental numbers of muons as a function of distance.