

Relativistic Acceleration Transformation

Consider Mary in a frame moving along the x axis with a velocity v relative to Frank. In the moving frame, a particle moving with a velocity u' has an acceleration a'. What does Frank observe?

The Lorentz transformations are:

$$x = \gamma(x' + vt') : y = y' : z = z' : t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$x' = \gamma(x - vt) : y' = y : z' = z : t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

Let me get the velocity transforms first.

$$x = \gamma(x' + vt') : y = y' : z = z' : t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \frac{dx}{dt} = u_x = \frac{\gamma(dx' + v dt')}{\frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1-\beta^2}}} = \left[\frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \right]$$

$$\Rightarrow \frac{dy}{dt} = u_y = \frac{dy'}{dt'} = \frac{dy'}{\frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1-\beta^2}}} = \sqrt{1-\beta^2} \frac{\frac{dy'}{dt'}}{1 + \frac{v}{c^2} \frac{dx'}{dt'}} = \sqrt{1-\beta^2} \frac{u'_y}{1 + \frac{v}{c^2} u'_x} = \frac{u'_y}{\gamma \left(1 + \frac{v}{c^2} u'_x \right)}$$

$$\Rightarrow \frac{dz}{dt} = u_z = \frac{u'_z}{\gamma \left(1 + \frac{v}{c^2} u'_x \right)}$$

Now to obtain the acceleration transformations, we take the differential increments again. The resulting transformations (as will be shown below) are given by:

$$a_x = \frac{a'_x}{\gamma^3 \left[1 + \frac{u'_x v}{c^2} \right]^3}$$

$$a_y = \frac{1}{\gamma^2 \left[1 + \frac{v}{c^2} u'_x \right]^2} \left[a'_y - a'_x \frac{u'_y}{\left[1 + \frac{v}{c^2} u'_x \right]} \frac{v}{c^2} \right]$$

$$a_z = \frac{1}{\gamma^2 \left[1 + \frac{v}{c^2} u'_x \right]^2} \left[a'_z - a'_x \frac{u'_z}{\left[1 + \frac{v}{c^2} u'_x \right]} \frac{v}{c^2} \right]$$

Derivation of acceleration transformations:

$$\begin{aligned}
 x &= \gamma(x' + vt') : y = y' : z = z' : t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}} \\
 \Rightarrow a_x &= \frac{u_x}{dt} = \frac{\left[\frac{du'_x}{1 + \frac{u'_x v}{c^2}} \right] \left[\frac{u'_x + v}{\left(1 + \frac{u'_x v}{c^2}\right)^2} \right] \frac{v}{c^2} du'_x}{\frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1-\beta^2}}} = \sqrt{1-\beta^2} \frac{\left[\frac{du'_x}{1 + \frac{u'_x v}{c^2}} \right] \left[\frac{u'_x + v}{\left(1 + \frac{u'_x v}{c^2}\right)^2} \right] \frac{v}{c^2} du'_x}{dt' + \frac{v}{c^2} dx'} \\
 &= \sqrt{1-\beta^2} \frac{\left[\frac{a'_x}{1 + \frac{u'_x v}{c^2}} \right] \left[\frac{u'_x + v}{\left(1 + \frac{u'_x v}{c^2}\right)^2} \right] \frac{v}{c^2} a'_x}{1 + \frac{v}{c^2} u'_x} = \frac{a'_x}{\left[1 + \frac{u'_x v}{c^2}\right]^2} \sqrt{1-\beta^2} \left[1 - \frac{v}{c^2} \frac{u'_x + v}{\left(1 + \frac{u'_x v}{c^2}\right)} \right] \\
 &= \frac{a'_x}{\left[1 + \frac{u'_x v}{c^2}\right]^2} \sqrt{1-\beta^2} \left[\frac{\left(1 + \frac{u'_x v}{c^2}\right)}{\left(1 + \frac{u'_x v}{c^2}\right)} - \frac{v}{c^2} \frac{u'_x + v}{\left(1 + \frac{u'_x v}{c^2}\right)} \right] \\
 &= \frac{a'_x}{\left[1 + \frac{u'_x v}{c^2}\right]^2} \sqrt{1-\beta^2} \left[\frac{\left(1 + \frac{u'_x v}{c^2} - \frac{v u'_x}{c^2} - \beta^2\right)}{\left(1 + \frac{u'_x v}{c^2}\right)} \right] = \frac{a'_x}{\left[1 + \frac{u'_x v}{c^2}\right]^3} [1-\beta^2]^{3/2} \\
 \Rightarrow a_x &= \frac{a'_x}{\gamma^3 \left[1 + \frac{u'_x v}{c^2}\right]^3} \\
 a_y &= \frac{du_y}{dt} = \frac{\frac{1}{\gamma} \frac{du'_y}{1 + \frac{v}{c^2} u'_x} - \frac{1}{\gamma} \frac{u'_y}{\left[1 + \frac{v}{c^2} u'_x\right]^2} \frac{v}{c^2} du'_x}{\frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1-\beta^2}}} = \frac{1}{\gamma} \sqrt{1-\beta^2} \frac{\frac{a'_y}{1 + \frac{v}{c^2} u'_x} - \frac{u'_y}{\left[1 + \frac{v}{c^2} u'_x\right]^2} \frac{v}{c^2} a'_x}{1 + \frac{v}{c^2} u'_x} \\
 &= \frac{1}{\gamma} \frac{\sqrt{1-\beta^2}}{\left[1 + \frac{v}{c^2} u'_x\right]^2} \left[a'_y - a'_x \frac{u'_y}{\left[1 + \frac{v}{c^2} u'_x\right]} \frac{v}{c^2} \right] = \frac{1}{\gamma^2 \left[1 + \frac{v}{c^2} u'_x\right]^2} \left[a'_y - a'_x \frac{u'_y}{\left[1 + \frac{v}{c^2} u'_x\right]} \frac{v}{c^2} \right] \\
 \Rightarrow a_z &= \frac{1}{\gamma^2 \left[1 + \frac{v}{c^2} u'_x\right]^2} \left[a'_z - a'_x \frac{u'_z}{\left[1 + \frac{v}{c^2} u'_x\right]} \frac{v}{c^2} \right]
 \end{aligned}$$

The transformations for the accelerations are not nearly as clean-cut as for the velocities, but one important point is shown here: where-as in the classical mechanics, accelerations are orthogonal along the three Cartesian coordinates when transformed, under special relativity this is not the case.

If the particle is moving along the x-axis only or if the acceleration is zero in the x-direction, these transformations reduce to become:

$$a_x = \frac{a'_x}{\gamma^3 \left[1 + \frac{u'_x v}{c^2}\right]^3} : a_y = \frac{a'_y}{\gamma^2 \left[1 + \frac{v}{c^2} u'_x\right]^2} : a_z = \frac{a'_z}{\gamma^2 \left[1 + \frac{v}{c^2} u'_x\right]^2}$$