Relativistic Doppler Shift

Suppose, that n the earth's frame of reference, a light pulse is sent out every ct. A spaceship moves towards the earth with a speed βc.

How often does the spaceship receive signals?

The important point here is that every observer measures the same speed of light. The clock in the moving space ship must, therefore, move slower than the earth-bound clock. An increment of time transforms as:

$$T' = \gamma T$$

where T is the time in a frame of reference which is not moving. If the spaceship is also not moving relative to the rest frame, the increments are the same.

Classically, the distance between the pulses is measured to be different in the two frames.

For a non-moving spaceship, the earth-bound observer measures the distance between successive pulses striking the spaceship to be $\lambda_0 = cT_0$ and $f_0 = \frac{c}{\lambda_0}$. This is the frequency that would be observed in the rest frame of the source. All observers stationary with respect to the source measure this same frequency.

For a moving spaceship, the earth-bound observer measures the total distance that the moving spaceship travels between pulses striking the spaceship $d = \beta c T_0$

This means that the earth-bound observer measures a total distance that the wavefront travels between successive pulses that strike the spaceship is:

$$\lambda = cT_0 - d = cT_0 - \beta c T_0 = cT_0(1-\beta) \Rightarrow \lambda = c\left(\frac{1}{f_0}\right)(1-\beta)$$

The earth-bound observer measures the pulses striking the spaceship with a frequency given by

$$f = \frac{c}{\lambda} = \frac{c}{c\left(\frac{1}{f_0}\right)(1-\beta)} = f_0 \frac{1}{1-\beta} = \frac{1+\beta}{(1-\beta)(1+\beta)} f_0 = \gamma^2 (1+\beta) f_0$$

To transform time increments from the earth-bound system to the spaceship system, we perform:

$$T' = \frac{1}{f'} = \gamma T$$

The proper time increment here is given by:

$$\frac{1}{T} = \gamma^2 (1+\beta) f_0 \Rightarrow T = \frac{1}{\gamma^2 (1+\beta) f_0}$$

which is exactly the reciprocal frequency that the earth-bound observer measures the pulses hitting the spaceship with.

Using the proper time increment to find the transformed time increments, we have

$$T' = \gamma T = \frac{1}{\gamma (1+\beta) f_0} \Rightarrow f' = \gamma (1+\beta) f_0 = \frac{1+\beta}{\sqrt{(1+\beta)(1-\beta)}} f_0 = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0$$

Now let's look at the case where the spaceship is moving away from the source.

For a moving spaceship, the earth-bound observer measures the total distance that the moving spaceship travels between pulses striking the spaceship

$$d = \beta c T_0$$

This means that the earth-bound observer measures a total distance that the wavefront travels between successive pulses that strike the spaceship is:

$$\lambda = cT_0 + d = cT_0 + \beta c T_0 = cT_0\left(1+\beta\right) \Rightarrow \lambda = c\left(\tfrac{1}{f_0}\right)\left(1+\beta\right)$$

The earth-bound observer measures the pulses striking the spaceship with a frequency given by

$$f = \tfrac{c}{\lambda} = \tfrac{c}{c\left(\frac{1}{f_0}\right)(1-\beta)} = f_0 \tfrac{1}{1-\beta} = \tfrac{1-\beta}{(1-\beta)(1+\beta)} f_0 = \gamma^2\left(1-\beta\right) f_0$$

To transform time increments from the earth-bound system to the spaceship system, we perform:

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The proper time increment here is given by:

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which is exactly the reciprocal frequency that the earth-bound observer measures the pulses hitting the spaceship with.

Using the proper time increment to find the transformed time increments, we have

$$T' = \gamma T = \tfrac{1}{\gamma(1-\beta) f_0} \Rightarrow f' = \gamma\left(1-\beta\right) f_0 = \tfrac{1-\beta}{\sqrt{(1+\beta)(1-\beta)}} f_0 = \tfrac{\sqrt{1-\beta}}{\sqrt{1+\beta}} f_0$$

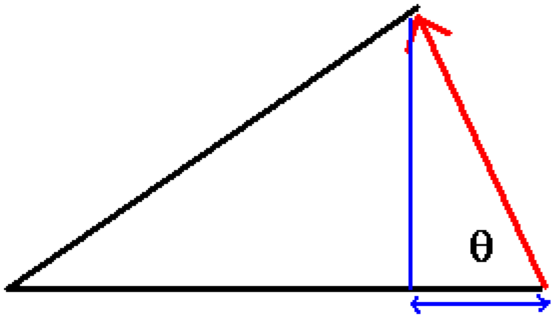
Now let's look at the case where the spaceship is moving away from the source.

These two equations can be combined if you let a +β represent a source and observer that are moving towards each other. If the source and observer are moving away from each other, then you use a -β. Under this agreement, the relativistic Doppler shift is:

$$f' = \tfrac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0$$

The relativistic Doppler shift is a very real phenomena and is observed with satellites, in astronomy and many other places.

Now let's see what happens when a source and observer are moving at some angle.



If the source is moving a bit towards the source, the wavelength is still shortened. By how much? The answer is shown in blue above:

$$d = \beta c T_0 \cos(\theta)$$

The wavelength is shortened by an amount as seen from an earth-bound observer:

$$\lambda = cT_0 - d = cT_0 - \beta c T_0 \cos(\theta) = cT_0\left(1-\beta\cos(\theta)\right) \Rightarrow \lambda = c\left(\tfrac{1}{f_0}\right)\left(1-\beta\cos(\theta)\right)$$

The earth-bound observer measures the pulses striking the spaceship with a frequency given by:

$$f = \tfrac{c}{\lambda} = \frac{c}{c\left(\frac{1}{f_0}\right)\left(1-\beta\cos(\theta)\right)} = f_0 \frac{1}{\left(1-\beta\cos(\theta)\right)}$$

Now time is dilated on the moving spaceship, as before. This is given by:

$$T' = \gamma T \Rightarrow \tfrac{1}{f'} = \gamma\tfrac{1}{f} \Rightarrow f' = \tfrac{1}{\gamma} f$$

So we can find the transformed frequency as:

$$f' = \tfrac{1}{\gamma} f = \tfrac{1}{\gamma}\frac{1}{\left(1-\beta\cos(\theta)\right)} f_0 = \frac{\sqrt{1-\beta^2}}{\left(1-\beta\cos(\theta)\right)} f_0$$

This is my result, which is not in agreement with your author. It is, however, in agreement with Special Relativity by A.P. French (Mit Introductory Physics Course, 1968), eq. 5-17.